

A Proof of Sendov's Conjecture and Related Extremal Problems

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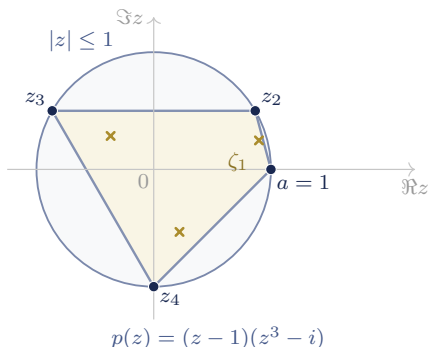
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Gauss–Lucas Theorem (1836, 1874)

Theorem (Gauss–Lucas)

Let $p \in \mathbb{C}[z]$ have degree $n \geq 2$, with zeros $z_1, \dots, z_n$. Then every critical point of p lies in the convex hull of the zeros of p .

That is, $p'(\zeta) = 0 \implies \zeta \in \text{conv}\{z_1, \dots, z_n\}$.



- Zeros of p
- × Critical points (zeros of p')
- Convex hull

Proof. Write

$$p(z) = c \prod_{j=1}^n (z - z_j).$$

Let $p'(\zeta) = 0$. If $p(\zeta) = 0$, the conclusion is immediate. Otherwise,

$$0 = \frac{p'(\zeta)}{p(\zeta)} = \sum_{j=1}^n \frac{1}{\zeta - z_j} = \sum_{j=1}^n \frac{\bar{\zeta} - \bar{z}_j}{|\zeta - z_j|^2}.$$

Taking complex conjugates and rearranging,

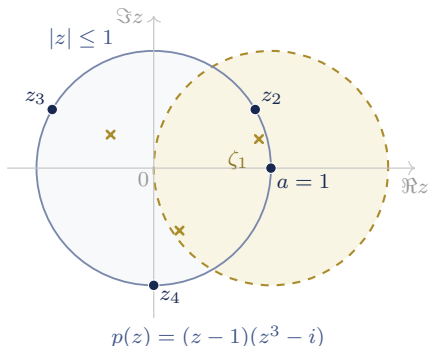
$$\boxed{\zeta = \sum_{j=1}^n \frac{|\zeta - z_j|^{-2}}{\sum_{k=1}^n |\zeta - z_k|^{-2}} z_j}.$$

Hence ζ is a convex combination of $z_1, \dots, z_n$. □

Sendov's Conjecture (1958)

Conjecture (Sendov)

Let $p \in \mathbb{C}[z]$ have degree $n \geq 2$, with all zeros in the closed unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. For every zero a of p , there exists a critical point ζ of p such that $|a - \zeta| \leq 1$.



- Zeros of p
- × Critical points (zeros of p')
- $|z-1| \leq 1$

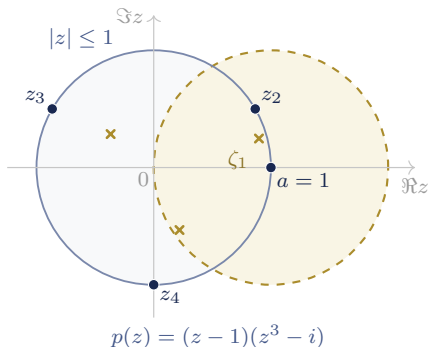
Progress by Degree

- $n = 3, 4$ (Rubinstein, 1968)
- $n \leq 5$ (Meir–Sharma, 1969)
- $n = 6$ (Brown, 1991)
- $n \leq 7$ (Borcea, 1996)
- $n \leq 8$ (Brown–Xiang, 1999)
- $n \geq n_0$ (Tao, 2020)
- $n \geq 10^{200000}$ (Zhang, 2026)
- **All** $n \geq 3$ (Mazur, 2026; Tao, digest)

Phelps–Rodriguez Conjecture (1972)

Conjecture (Phelps–Rodriguez)

Let $p \in \mathbb{C}[z]$ have degree $n \geq 2$, with all zeros in the closed unit disk $\overline{\mathbb{D}}$. For every zero a of p , there exists a critical point ζ of p such that $|a - \zeta| < 1$, unless $p = c(z^n - a^n)$ and $|a| = 1$, where $c \in \mathbb{C}$.

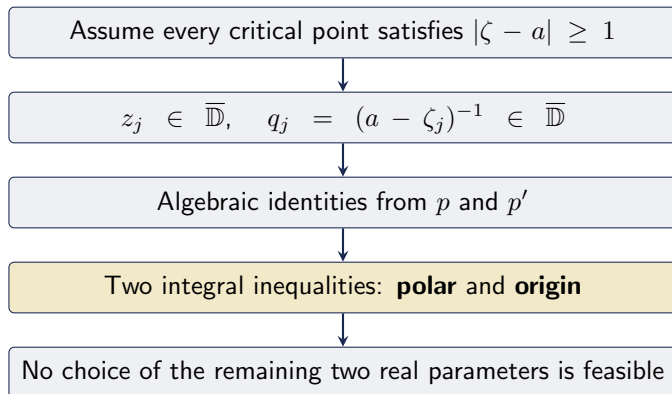


- Zeros of p
- × Critical points (zeros of p')
- $|z-1| \leq 1$

The Proof at a Glance

The Interior Case of Sendov's Conjecture

For every zero $a \in [0, 1)$ of p , there exists a critical point ζ such that $|\zeta - a| < 1$.



Proof due to Lech Mazur (2026), reorganized and streamlined in Tao's exposition.

Step 1. Normalize a Hypothetical Counterexample

By rotation, take $a \in [0, 1)$. Write the zeros as

$$p(z) = (z - a) \prod_{j=1}^{n-1} (z - z_j), \quad |z_j| \leq 1.$$

If $a = 0$, comparing the factorizations of $p'(0)$ gives $n \prod_{j=1}^{n-1} |\zeta_j| = \prod_{j=1}^{n-1} |z_j| \leq 1$, so some $|\zeta_j| < 1$. Henceforth $0 < a < 1$. Assume **no** critical point lies in the open disk $D(a, 1)$: $|a - \zeta_j| \geq 1$. Write the critical points as

$$\zeta_j = a - \frac{1}{q_j}, \quad q_j = \frac{1}{a - \zeta_j}, \quad |q_j| \leq 1.$$

Thus, we now have two collections of points:

$$\underbrace{z_1, \dots, z_{n-1}}_{\substack{\text{Remaining zeros} \\ \text{of } p}} \quad \text{and} \quad \underbrace{q_1, \dots, q_{n-1}}_{\substack{\text{Transformed} \\ \text{critical points}}}.$$

Can two collections of points in the closed unit disk simultaneously satisfy all the algebraic relations imposed by a polynomial and its derivative?

Step 2. Two Ways to Describe $p(z)$ and $p'(z)$

Normalize p to be monic, and set $N = n - 1$. We have

$$p(z) = \underbrace{(z - a) \prod_{j=1}^N (z - z_j)}_{\text{In terms of } z_1, \dots, z_N} = \underbrace{n(z - a) \int_0^1 \prod_{j=1}^N \left(t(z - a) + \frac{1}{q_j} \right) dt}_{\text{In terms of } q_1, \dots, q_N}.$$

Similarly, differentiating the two representations yields

$$p'(z) = \underbrace{\prod_{j=1}^N (z - z_j) + (z - a) \sum_{k=1}^N \prod_{\substack{j=1 \\ j \neq k}}^N (z - z_j)}_{\text{In terms of } z_1, \dots, z_N} = \underbrace{n \prod_{j=1}^N \left(z - a + \frac{1}{q_j} \right)}_{\text{In terms of } q_1, \dots, q_N}.$$

Recall that

$$p(a) = 0, \quad p'(z) = n \prod_{j=1}^N \left(z - a + \frac{1}{q_j} \right).$$

By the Newton–Leibniz formula,

$$p(z) - p(a) = \int_a^z p'(w) dw.$$

Parameterize the line segment from a to z by $w = a + t(z - a)$, $0 \leq t \leq 1$, so that $dw = (z - a) dt$. Consequently,

$$\begin{aligned} p(z) &= \int_a^z p'(w) dw = (z - a) \int_0^1 p'(a + t(z - a)) dt \\ &= n(z - a) \int_0^1 \prod_{j=1}^N \left(t(z - a) + \frac{1}{q_j} \right) dt. \end{aligned}$$

Step 3a. Polar Integral Identity

Centroid Identity (comparing coefficients).

$$\frac{a + \sum_{j=1}^N z_j}{n} = \frac{1}{N} \sum_{j=1}^N \left(a - \frac{1}{q_j} \right)$$

The mean of the zeros of p equals the mean of the zeros of p' .

Polar Integral Identity (Comparing $p(1/a)/p'(a)$)

$$\prod_{j=1}^N \frac{1 - az_j}{a - z_j} = \int_0^1 \prod_{j=1}^N (a + t(1 - a^2)q_j) dt.$$

For any $|z_j| \leq 1$,

$$|1 - az_j|^2 - |a - z_j|^2 = (1 - a^2)(1 - |z_j|^2) \geq 0.$$

Thus the **modulus of the left side is at least one**.

Step 3b. Origin Integral Identity

Identity 1 Relating $F(1)$ and $\int_0^1 F(t) dt$

Define the polynomial in t :

$$F(t) = \prod_{j=1}^N (1 - atq_j).$$

Evaluating p and p' at 0 gives, respectively,

$$(-1)^N \prod_{j=1}^N z_j = \frac{n}{\prod_j q_j} \int_0^1 F(t) dt,$$

$$(-1)^N \left(\prod_{j=1}^N z_j \right) \left(1 + a \sum_{j=1}^N \frac{1}{z_j} \right) = \frac{n}{\prod_j q_j} F(1).$$

Identity 1 Relating $F(1)$ and $\int_0^1 F(t) dt$

$$F(1) = \left(1 + a \sum_{j=1}^N \frac{1}{z_j} \right) \int_0^1 F(t) dt, \quad \int_0^1 F(t) dt = \frac{(-1)^N}{n} \prod_{j=1}^N z_j q_j.$$

Step 3b. Origin Integral Identity

Eliminating $F(1)$: Finding Identity 2 Relating $F(1)$ and $\int_0^1 F(t) dt$

Recall that

$$F(t) = \prod_{j=1}^N (1 - atq_j), \quad F(0) = 1.$$

For any constant $C \in \mathbb{C}$, the Newton–Leibniz formula gives

$$1 = F(1) + C \int_0^1 F(t) dt - \int_0^1 (F'(t) + CF(t)) dt.$$

How should we choose C ?

We want the remainder

$$R_C(t) := F'(t) + CF(t)$$

to vanish at $t = 0$. Since $R_C(t)$ is a polynomial, this ensures that

$$R_C(t) = tG(t)$$

for some polynomial G . The factor t is crucial for deriving the subsequent origin inequality.

Since $F(0) = 1$ and $F'(0) = -a \sum_{j=1}^N q_j$, the natural choice is

$$C = -\frac{F'(0)}{F(0)} = a \sum_{j=1}^N q_j = Na\mu, \quad \mu = \frac{1}{N} \sum_{j=1}^N q_j.$$

Direct differentiation now gives

$$F'(t) + Na\mu F(t) = -a^2 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k).$$

Integrating the preceding identity over $[0, 1]$ and using $F(0) = 1$, we obtain

Identity 2 Relating $F(1)$ and $\int_0^1 F(t) dt$

$$1 = F(1) + Na\mu \int_0^1 F(t) dt + a^2 \int_0^1 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k) dt.$$

Step 3b. Origin Integral Identity

Recall Identities 1 and 2:

$$F(1) = \left(1 + a \sum_{j=1}^N \frac{1}{z_j}\right) \int_0^1 F(t) dt.$$

$$1 = F(1) + Na\mu \int_0^1 F(t) dt + a^2 \int_0^1 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k) dt.$$

Combining the two origin integral identities and eliminating $F(1)$, we obtain

Origin Integral Identity

$$1 = \left(1 + a \sum_{j=1}^N \frac{1}{z_j} + Na\mu\right) \int_0^1 F(t) dt + a^2 \int_0^1 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k) dt.$$

Step 4a. From the Polar Integral Identity to the Polar Inequality

Recall the polar integral identity:

$$\prod_{j=1}^N \frac{1 - az_j}{a - z_j} = \int_0^1 \prod_{j=1}^N (a + t(1 - a^2)q_j) dt.$$

Since $|z_j| \leq 1$,

$$|1 - az_j|^2 - |a - z_j|^2 = (1 - a^2)(1 - |z_j|^2) \geq 0.$$

Taking absolute values and applying the triangle inequality,

$$1 \leq \prod_{j=1}^N \left| \frac{1 - az_j}{a - z_j} \right| \leq \int_0^1 \prod_{j=1}^N |a + t(1 - a^2)q_j| dt.$$

By the power-mean inequality and $|q_j| \leq 1$,

$$\begin{aligned}
 1 &\leq \prod_{j=1}^N \left| \frac{1 - az_j}{a - z_j} \right| \leq \int_0^1 \prod_{j=1}^N |a + t(1 - a^2)q_j| dt \leq \int_0^1 \left(\frac{1}{N} \sum_{j=1}^N |a + t(1 - a^2)q_j|^2 \right)^{N/2} dt \\
 &= \int_0^1 \left(a^2 + 2at(1 - a^2)x + \frac{t^2(1 - a^2)^2}{N} \sum_{j=1}^N |q_j|^2 \right)^{N/2} dt \quad (\text{Set } \mu = x + iy) \\
 &\leq \int_0^1 (a^2 + 2at(1 - a^2)x + t^2(1 - a^2)^2)^{N/2} dt \\
 &= \int_0^1 (1 + (1 - a^2)[-1 + (2 - \beta(1))t] - (1 - a^2)^2 t(1 - t))^{N/2} dt \quad (\text{Set } \beta(1) = 1 - 2ax + a^2) \\
 &< \int_0^1 (1 + (1 - a^2)[-1 + (2 - \beta(1))t])^{N/2} dt \quad (\text{Set } \alpha = \frac{N}{2}(1 - a^2)) \\
 &\leq \int_0^1 e^{\alpha[-1 + (2 - \beta(1))t]} dt. \quad (\text{By } 1 + s \leq e^s.)
 \end{aligned}$$

Recall that

$$1 < \int_0^1 e^{\alpha[-1+(2-\beta(1))t]} dt = \frac{e^{\alpha(1-\beta(1))} - e^{-\alpha}}{\alpha(2-\beta(1))},$$

where $\beta(1) = 1 - 2ax + a^2$ and $\alpha = \frac{N}{2}(1 - a^2)$.

Claim 1. $0 < \beta(1) < 1$. In particular, $x > a/2 > 0$.

Since $\beta(1) \geq (1 - a)^2 > 0$, it remains to prove $\beta(1) < 1$. If $\beta(1) \geq 1$, then $-1 + (2 - \beta(1))t \leq 0$ for all $t \in [0, 1]$, so the preceding integral is at most 1, a contradiction.

Claim 2. $\beta(1) < 1 - \frac{\log \alpha}{\alpha}$.

Notice that

$$e^{\alpha(1-\beta(1))} > \alpha(2 - \beta(1)) + e^{-\alpha} > \alpha.$$

Taking logarithms proves Claim 2.

Step 4a. Polar Inequality

Claim 3. $\beta(1) < \frac{\alpha}{\alpha+3}$.

Using the inequality $\log(\sinh t/t) \leq \sqrt{t^2+9}-3$, we obtain

$$\begin{aligned} 1 &< \frac{e^{\alpha(1-\beta(1))} - e^{-\alpha}}{\alpha(2-\beta(1))} \\ &= e^{-\alpha\beta(1)/2} \frac{\sinh(\alpha(1-\beta(1)/2))}{\alpha(1-\beta(1)/2)} \\ &\leq \exp\left(-\frac{\alpha\beta(1)}{2} + \sqrt{9 + \alpha^2(1-\beta(1)/2)^2} - 3\right). \end{aligned}$$

Consequently,

$$\begin{aligned} 3 + \frac{\alpha\beta(1)}{2} &< \sqrt{9 + \alpha^2(1-\beta(1)/2)^2} \implies 3\beta(1) < \alpha(1-\beta(1)) \\ &\implies \beta(1) < \frac{\alpha}{\alpha+3}. \end{aligned}$$

Polar Inequality

$$0 < \beta(1) < \min\left\{\frac{\alpha}{\alpha+3}, 1 - \frac{\log \alpha}{\alpha}\right\}.$$

Step 4b. Origin Integral Identity

Separate the Two Terms

Assume $n \geq 5$; recall $N = n - 1$, $\mu = x + iy$, and

$$F(t) = \prod_{j=1}^N (1 - atq_j), \quad J = \prod_{j=1}^N z_j q_j, \quad \int_0^1 F(t) dt = \frac{(-1)^N}{n} J.$$

The origin identity is

$$1 = \left(1 + a \sum_{j=1}^N \frac{1}{z_j} + Na\mu \right) \int_0^1 F(t) dt + a^2 \int_0^1 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k) dt.$$

Put

$$T_1 := \left| \left(1 + a \sum_{j=1}^N \frac{1}{z_j} + Na\mu \right) \int_0^1 F(t) dt \right|,$$

$$T_2 := a^2 \left| \int_0^1 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k) dt \right|.$$

$$1 \leq T_1 + T_2.$$

If $z_j = 0$, products like J/z_j mean $q_j \prod_{k \neq j} z_k q_k$; all the relevant expressions are interpreted by continuous extension.

Step 4b. First Term T_1

By the centroid identity,

$$\sum_{j=1}^N \overline{z_j} = Na - \frac{n}{N} \sum_{j=1}^N \frac{1}{q_j}.$$

$$\begin{aligned} T_1 &= \frac{1}{n} \left| J \left(1 + a \sum_{j=1}^N \frac{1}{z_j} + Na\mu \right) \right| \\ &= \frac{1}{n} \left| J \left[1 + a \sum_{j=1}^N \overline{z_j} + Na\mu + a \sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z_j} \right) \right] \right| && \text{(add and subtract } \overline{z_j} \text{)} \\ &= \frac{1}{n} \left| J \left[1 + Na^2 - \frac{an}{N} \sum_{j=1}^N \frac{1}{q_j} + Na\mu + a \sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z_j} \right) \right] \right| && \text{(centroid identity)} \\ &= \frac{1}{n} \left| (1 + Na^2 - a\mu)J + aJ \left[\sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z_j} \right) - \frac{n}{N} \sum_{j=1}^N \left(\frac{1}{q_j} - q_j \right) \right] \right| && \text{(add and subtract } q_j \text{).} \end{aligned}$$

Step 4b. First Term T_1

Write $A := 1 + Na^2 - a\mu$ and recall $|J| \leq 1$.

$$\begin{aligned} T_1 &= \frac{1}{n} \left| AJ + aJ \left[\sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z_j} \right) - \frac{n}{N} \sum_{j=1}^N \left(\frac{1}{\overline{q_j}} - q_j \right) \right] \right| \\ &\leq \frac{|J|}{n} |A| + \frac{a|J|}{n} \left[\sum_{j=1}^N \left| \frac{1}{z_j} - \overline{z_j} \right| + \frac{n}{N} \sum_{j=1}^N \left| \frac{1}{\overline{q_j}} - q_j \right| \right] && \text{(triangle inequality)} \\ &\leq \frac{|J|}{n} |A| + \frac{a}{N} (1 - |J|^2) && \text{(defect lemma; } n/N \geq 1) \\ &\leq \frac{|A|}{n} && (|A|/n \geq 2a/N \text{ and } 0 \leq |J| \leq 1) \\ &\leq \frac{1 + Na^2 - ax}{n} + \frac{1 - x^2}{2nN} && (x^2 + y^2 \leq 1). \end{aligned}$$

$$T_1 \text{ Bound: } \boxed{T_1 \leq \frac{1 + Na^2 - ax}{n} + \frac{1 - x^2}{2nN}.$$

The Defect Lemma

Defect Lemma. For $w_1, \dots, w_m \in \overline{\mathbb{D}}$,

$$\sum_{j=1}^m (1 - |w_j|^2) \prod_{k \neq j} |w_k| \leq 1 - \prod_{j=1}^m |w_j|^2.$$

Proof. Let $r_j = |w_j|$, and let S_m denote the sum on the left. The case $m = 1$ is immediate. Set $R = \prod_{j=1}^{m-1} r_j$. By induction,

$$\begin{aligned} S_m &= r_m S_{m-1} + R(1 - r_m^2) \\ &\leq r_m(1 - R^2) + R(1 - r_m^2) \\ &= 1 - R^2 r_m^2 - (1 - R)(1 - r_m)(1 - R r_m) \\ &\leq 1 - R^2 r_m^2. \quad \square \end{aligned}$$

Apply the lemma to $z_1, \dots, z_N, \overline{q_1}, \dots, \overline{q_N}$ to obtain

$$|J| \left[\sum_{j=1}^N \left| \frac{1}{z_j} - \overline{z_j} \right| + \sum_{j=1}^N \left| \frac{1}{\overline{q_j}} - q_j \right| \right] \leq 1 - |J|^2.$$

Controlling the Modulus

For $N = n - 1 \geq 4$ and $|\mu| \leq 1$,

$$\frac{|A|}{n} \geq \frac{1 + Na^2 - a}{n} \geq \frac{2a}{N}, \quad A := 1 + Na^2 - a\mu.$$

Indeed, the last inequality is equivalent to

$$N^2a^2 - (3n - 1)a + N \geq 0,$$

which follows from $N^2a^2 + N \geq 2N^{3/2}a \geq (3N + 2)a$. Therefore

$$\begin{aligned} \frac{|J|}{n}|A| + \frac{a}{N}(1 - |J|^2) &\leq \frac{|A|}{n} \left(|J| + \frac{1 - |J|^2}{2} \right) \\ &\leq \frac{|A|}{n}. \end{aligned}$$

Next, put $u = 1 + Na^2 - ax \geq Na^2 > 0$. Since $x^2 + y^2 \leq 1$,

$$|A| = \sqrt{u^2 + a^2y^2} \leq u + \frac{a^2y^2}{2u} \leq 1 + Na^2 - ax + \frac{1 - x^2}{2N}.$$

Step 4b. Second Term T_2

A Continuous Chain of Estimates (I)

Recall $|q_j| \leq 1$. We begin by taking absolute values:

$$\begin{aligned} T_2 &= a^2 \left| \int_0^1 t \sum_{j=1}^N q_j^2 \prod_{k \neq j} (1 - atq_k) dt \right| \\ &\leq a^2 \int_0^1 t \sum_{j=1}^N |q_j|^2 \prod_{k \neq j} |1 - atq_k| dt && \text{(triangle inequality)} \\ &\leq a^2 \int_0^1 t \sum_{j=1}^N \prod_{k \neq j} |1 - atq_k| dt && (|q_j| \leq 1). \end{aligned}$$

Step 4b. Second Term T_2

Put $\beta(t) = 1 - 2atx + a^2t^2$ and $N = n - 1$.

$$\begin{aligned} T_2 &\leq a^2 \int_0^1 t \sum_{j=1}^N \prod_{k \neq j} |1 - atq_k| dt \\ &\leq a^2 N \int_0^1 t \left(\frac{1}{N} \sum_{j=1}^N |1 - atq_j| \right)^{N-1} dt && \text{(Maclaurin)} \\ &\leq a^2 N \int_0^1 t \left(\frac{1}{N} \sum_{j=1}^N |1 - atq_j|^2 \right)^{(N-1)/2} dt && \text{(power mean)} \\ &= a^2 N \int_0^1 t \left(1 - 2atx + \frac{a^2t^2}{N} \sum_{j=1}^N |q_j|^2 \right)^{(N-1)/2} dt \\ &\leq a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt && (|q_j| \leq 1). \end{aligned}$$

Step 4b. Second Term T_2

From the polar inequality, $x > a/2 > 0$; thus $0 < ax < 1$.

$$\begin{aligned} T_2 &\leq a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt \\ &\leq a^2 N \int_0^1 t(1-axt)^{n-2} dt + \frac{a^4 N(n-2)(1-x^2)}{2} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt \quad (\text{mean value thm}) \\ &\leq \frac{1}{nx^2} + \frac{a^4 N(n-2)(1-x^2)}{2} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt \quad (\text{beta integral}). \end{aligned}$$

T_2 **Bound:**

$$T_2 \leq a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt \leq \frac{1}{nx^2} + \frac{a^4 N(n-2)(1-x^2)}{2} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt.$$

Maclaurin and the Power-Mean Inequality

For $v_j := |1 - atq_j| \geq 0$, Maclaurin's inequality gives

$$\sum_{j=1}^N \prod_{k \neq j} v_k \leq N \left(\frac{1}{N} \sum_{j=1}^N v_j \right)^{N-1}.$$

Next, by the quadratic-mean inequality,

$$\begin{aligned} \left(\frac{1}{N} \sum_{j=1}^N v_j \right)^{N-1} &\leq \left(\frac{1}{N} \sum_{j=1}^N v_j^2 \right)^{(N-1)/2} \\ &= \left(1 - 2atx + \frac{a^2 t^2}{N} \sum_{j=1}^N |q_j|^2 \right)^{(N-1)/2} \\ &\leq \beta(t)^{(N-1)/2}. \end{aligned}$$

Here $\frac{1}{N} \sum_j q_j = \mu = x + iy$ and $|q_j| \leq 1$.

Mean Value Theorem and Beta Integral

The exact decomposition is

$$\beta(t) = (1 - axt)^2 + a^2 t^2 (1 - x^2).$$

Set $m = (n - 2)/2 \geq 1$. Since $1 - axt > 0$, the mean value theorem yields

$$\begin{aligned}\beta(t)^m - (1 - axt)^{2m} &\leq m[\beta(t) - (1 - axt)^2]\beta(t)^{m-1} \\ &= \frac{a^2(n-2)}{2}t^2(1-x^2)\beta(t)^{(n-4)/2}.\end{aligned}$$

Moreover, substituting $s = axt$ and extending the interval gives

$$\begin{aligned}\int_0^1 t(1 - axt)^{n-2} dt &\leq \frac{1}{a^2 x^2} \int_0^1 s(1 - s)^{n-2} ds \\ &= \frac{1}{a^2 x^2 n(n-1)} = \frac{1}{a^2 x^2 nN}.\end{aligned}$$

Step 4b. Origin Inequality

Combine the Two Termwise Estimates

Recall $1 \leq T_1 + T_2$ and

$$T_1 \leq \frac{1 + Na^2 - ax}{n} + \frac{1 - x^2}{2nN}, \quad 1 + Na^2 = n - 2\alpha, \quad \alpha = \frac{N}{2}(1 - a^2).$$

Also,

$$T_2 \leq a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt \leq \frac{1}{nx^2} + \frac{a^4 N(n-2)(1-x^2)}{2} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt.$$

Origin Inequalities

$$\begin{aligned} 1 &\leq \frac{1 + Na^2 - ax}{n} + \frac{1 - x^2}{2nN} + a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt \\ &\leq \frac{1 + Na^2 - ax}{n} + \frac{1 - x^2}{2nN} + \frac{1}{nx^2} + \frac{a^4 N(n-2)(1-x^2)}{2} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt. \end{aligned}$$

Claim: $\alpha \leq 17$

Claim. $\alpha \leq 17$ for all $n \geq 2$.

Proof. Suppose that $\alpha > 17$. Then $n \geq 36$. Set $u = a^2/(1 - a^2)$.

$$1 \leq \frac{1 + Na^2 - ax}{n} + \frac{1 - x^2}{2nN} + a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt \quad (\text{Origin inequality 1})$$

$$\leq 1 - \frac{2\alpha}{n} + \frac{1}{2nN} + a^2 N \int_0^1 t \beta(t)^{(n-2)/2} dt \quad (x > 0, 1 - x^2 \leq 1)$$

$$\leq 1 - \frac{2\alpha}{n} + \frac{1}{2nN} + \frac{4N}{(n-2)^2 x^2} + \frac{a^2 N}{2} \beta(1)^{(n-2)/2} \quad (\text{Estimate 1})$$

$$< 1 + \frac{\alpha}{n} \left[-2 + \frac{1}{2N\alpha} + \frac{4nN}{(n-2)^2 \log \alpha} + \frac{2n}{N} u(1+u)\alpha^{-u+1/(2\alpha)} \right] \quad (\text{Estimate 2})$$

$$< 1 + \frac{\alpha}{n} \left[-2 + \frac{1}{2 \cdot 35 \cdot 17} + \frac{4 \cdot 36 \cdot 35}{34^2 \log 17} + \frac{72}{35} 17^{1/34} (0.1825) \right] \quad (\text{Estimate 3})$$

$$< 1 + \frac{\alpha}{n} (1.948 - 2) < 1 \quad (\text{Numerical estimate}).$$

This is a contradiction. □

Proof of the Claim: $\alpha \leq 17$

Estimate 1. We have

$$\int_0^1 t\beta(t)^{(n-2)/2} dt \leq \frac{4}{(n-2)^2 a^2 x^2} + \frac{1}{2}\beta(1)^{(n-2)/2}.$$

Proof. Recall that

$$\beta(t) = 1 - 2axt + a^2 t^2, \quad \beta'(t) = 2a(at - x).$$

Since $a^2 t^2 \leq axt$ for $t \leq x/a$, and β is increasing on $[x/a, \infty)$,

$$\beta(t) \leq \begin{cases} 1 - axt \leq e^{-axt}, & 0 \leq t \leq \min\{1, x/a\}, \\ \beta(1), & x/a \leq t \leq 1. \end{cases}$$

The second interval is omitted if $x/a > 1$.

Consequently,

$$\begin{aligned} \int_0^1 t\beta(t)^{(n-2)/2} dt &\leq \int_0^\infty te^{-(n-2)axt/2} dt + \beta(1)^{(n-2)/2} \int_0^1 t dt \\ &= \frac{4}{(n-2)^2 a^2 x^2} + \frac{1}{2}\beta(1)^{(n-2)/2}. \end{aligned}$$



Proof of the Claim: $\alpha \leq 17$

Estimate 2. Under the assumption $\alpha > 17$, we have

$$x^2 > \frac{\log \alpha}{\alpha}, \quad \beta(1) < \alpha^{-1/\alpha}.$$

Proof. Recall the logarithmic polar bound:

$$\beta(1) < 1 - \frac{\log \alpha}{\alpha}.$$

On the other hand,

$$\begin{aligned} \beta(1) &= 1 - 2ax + a^2 = 1 - x^2 + (a - x)^2 \\ &\geq 1 - x^2. \end{aligned}$$

Therefore,

$$\begin{aligned} x^2 &\geq 1 - \beta(1) > \frac{\log \alpha}{\alpha}, \\ \beta(1) &< 1 - \frac{\log \alpha}{\alpha} \leq \exp\left(-\frac{\log \alpha}{\alpha}\right) = \alpha^{-1/\alpha}. \end{aligned}$$



Proof of the Claim: $\alpha \leq 17$

Estimate 3. We have

$$\sup_{u \geq 0} u(1+u)17^{-u} < 0.1825.$$

Proof. Put $k = \log 17$ and define

$$f(u) := u(1+u)e^{-ku}, \quad u \geq 0.$$

Differentiating gives

$$f'(u) = e^{-ku}(1 + 2u - ku(1+u)).$$

Thus $f'(u) = 0$ if and only if

$$ku^2 + (k-2)u - 1 = 0,$$

whose unique positive root is

$$u_* = \frac{2 - k + \sqrt{k^2 + 4}}{2k}.$$

Since $f(0) = \lim_{u \rightarrow \infty} f(u) = 0$ and f' changes sign from positive to negative at u_* , the global maximum is $f(u_*)$.

Direct numerical bounds give

$$2.83321 < k < 2.83322, \quad 0.46498 < u_* < 0.46499.$$

Consequently,

$$\begin{aligned} \sup_{u \geq 0} f(u) &= f(u_*) < 0.46499(1 + 0.46499)e^{-2.83321(0.46498)} \\ &< 0.18246 < 0.1825. \end{aligned}$$



Combining the Origin and Polar Inequalities

Let $n \geq 5$. Recall $N = n - 1$, $0 < \alpha \leq 17$,

$$ax = 1 - \frac{\alpha}{N} - \frac{\beta(1)}{2}, \quad a^2 = 1 - \frac{2\alpha}{N}.$$

Define

$$\begin{aligned} c(\alpha) &:= 1 - \frac{\alpha}{N} - \frac{\alpha}{2(3 + \alpha)}, \\ Q(t) &:= 1 - 2c(\alpha)t + a^2t^2. \end{aligned}$$

Then the refined origin bound gives

$$1 \leq \frac{1-x^2}{4\alpha N} + \frac{x^{-2}-1}{2\alpha} + \frac{1}{2N} + \frac{\beta(1)}{4\alpha} + \frac{a^4 n N (n-2)(1-x^2)}{4\alpha} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt \quad (\text{Origin Inequality 2})$$

$$\leq \frac{\beta(1)}{2\alpha(1-\beta(1))} + \frac{\beta(1)}{4\alpha} + \frac{1}{2N} + \frac{\beta(1)}{4\alpha N} + \frac{a^4 n N (n-2)\beta(1)}{4\alpha} \int_0^1 t^3 \beta(t)^{(n-4)/2} dt \quad (\text{Estimate 1})$$

$$< \frac{1}{6} + \frac{1}{4(3+\alpha)} + \frac{1}{2N} + \frac{1}{4N(3+\alpha)} + \frac{a^4 n N (n-2)}{4(3+\alpha)} \int_0^1 t^3 Q(t)^{(n-4)/2} dt =: R_n(\alpha) \quad (\text{Estimate 2}).$$

A Necessary Inequality for a Counterexample

A Necessary Inequality for a Counterexample

For every $n \geq 5$, we have

$$1 \leq R_n(\alpha) := \frac{1}{6} + \frac{1}{4(3+\alpha)} + \frac{1}{2N} + \frac{1}{4N(3+\alpha)} + \frac{a^4 n N (n-2)}{4(3+\alpha)} \int_0^1 t^3 Q(t)^{(n-4)/2} dt.$$

For $n \geq 201$ and $0 < \alpha \leq 17$, we continue the same upper-bound chain:

$$\begin{aligned} 1 &\leq R_n(\alpha) \\ &\leq \frac{1}{6} + \frac{1}{4(3+\alpha)} + \frac{1}{2N} + \frac{1}{4N(3+\alpha)} + \frac{a^4 n N (n-2)}{4(3+\alpha)} \left[\frac{96}{(n-4)^4 c(\alpha)^4} + \frac{1}{4} \left(\frac{\alpha}{3+\alpha} \right)^{(n-4)/2} \right] \end{aligned} \quad (\text{Estimate 3})$$

$$\leq \frac{1}{4} + \frac{7}{12N} + \frac{24nN(n-2)}{(n-4)^4} + \frac{nN(n-2)}{48} \left(\frac{17}{20} \right)^{(n-4)/2} \quad (\text{Estimate 4})$$

$$\therefore H_n \leq H_{201} < 0.399 < 1 \quad (\text{Estimate 5}).$$

Case I

Sendov's conjecture holds for all $n \geq 201$.

The estimates used in the chain are proved on the next slides.

Estimate 1: Rewrite the Origin Bound

Estimate 1. We have

$$1 - x^2 \leq \beta(1), \quad \frac{1}{x^2} - 1 \leq \frac{\beta(1)}{1 - \beta(1)}, \quad ax = 1 - \frac{\alpha}{N} - \frac{\beta(1)}{2}.$$

Proof. Since $\beta(1) = 1 - 2ax + a^2$,

$$\beta(1) = 1 - x^2 + (a - x)^2 \geq 1 - x^2.$$

The polar bound gives $0 < \beta(1) < 1$, hence $x^2 \geq 1 - \beta(1) > 0$ and

$$\frac{1}{x^2} - 1 \leq \frac{1}{1 - \beta(1)} - 1 = \frac{\beta(1)}{1 - \beta(1)}.$$

Finally, $a^2 = 1 - 2\alpha/N$ gives

$$2ax = 1 + a^2 - \beta(1) = 2 - \frac{2\alpha}{N} - \beta(1).$$



Estimate 2: The Rational Polar Bound

Estimate 2. Recall $\beta(1) < \alpha/(3 + \alpha)$. Since

$$ax = 1 - \frac{\alpha}{N} - \frac{\beta(1)}{2} > c(\alpha),$$

we obtain

$$\begin{aligned} 0 < c(\alpha) < ax \leq a, \\ \beta(t) = 1 - 2axt + a^2t^2 \leq Q(t) \quad (0 \leq t \leq 1), \\ Q(1) = 1 - 2c(\alpha) + a^2 = \frac{\alpha}{3 + \alpha}. \end{aligned}$$

(The positivity of $c(\alpha)$ is verified in Estimate 4.) Since $s/(1 - s)$ is increasing on $(0, 1)$,

$$\begin{aligned} \frac{\beta(1)}{2\alpha(1 - \beta(1))} < \frac{1}{6}, \quad \frac{\beta(1)}{4\alpha} < \frac{1}{4(3 + \alpha)}, \\ \frac{\beta(1)}{4\alpha N} < \frac{1}{4N(3 + \alpha)}. \end{aligned}$$

Since $(n - 4)/2 \geq 0$ and $\beta(t) \leq Q(t)$, the integral bound follows. □

Estimate 3: The Last Integral

Estimate 3. Put $c = c(\alpha) > 0$ and $s = c/a^2$. The quadratic $Q(t) = 1 - 2ct + a^2t^2$ decreases up to $t = s$ and increases afterwards. Consequently,

$$Q(t) \leq \begin{cases} 1 - ct \leq e^{-ct}, & 0 \leq t \leq \min\{s, 1\}, \\ Q(1) = \frac{\alpha}{3 + \alpha}, & s \leq t \leq 1. \end{cases}$$

The second interval is omitted when $s > 1$. Writing $m = (n - 4)/2 > 0$, we obtain

$$\begin{aligned} \int_0^1 t^3 Q(t)^m dt &\leq \int_0^\infty t^3 e^{-mct} dt + \frac{1}{4} \left(\frac{\alpha}{3 + \alpha} \right)^m \\ &= \frac{6}{(mc)^4} + \frac{1}{4} \left(\frac{\alpha}{3 + \alpha} \right)^m. \quad \square \end{aligned}$$

Thus, we obtain the integral estimate:

$$\int_0^1 t^3 Q(t)^{(n-4)/2} dt \leq \frac{96}{(n-4)^4 c(\alpha)^4} + \frac{1}{4} \left(\frac{\alpha}{3 + \alpha} \right)^{(n-4)/2}.$$

Estimate 4: A Uniform Bound for $c(\alpha)$

Estimate 4. For $n \geq 201$, $N \geq 200$, and $0 < \alpha \leq 17$, put

$$c_{201}(\alpha) := 1 - \frac{\alpha}{200} - \frac{\alpha}{2(3 + \alpha)}.$$

Then $c(\alpha) \geq c_{201}(\alpha) \geq 49/100 > 0$. Direct differentiation gives

$$\begin{aligned} \frac{d}{d\alpha} [(3 + \alpha)c_{201}(\alpha)^4] \\ = -\frac{c_{201}(\alpha)^3}{200(3 + \alpha)} [5(3 + \alpha)^2 - 103(3 + \alpha) + 900] < 0. \end{aligned}$$

Indeed, the quadratic in brackets has discriminant $-7391 < 0$. Therefore

$$\begin{aligned} (3 + \alpha)c(\alpha)^4 &\geq (3 + \alpha)c_{201}(\alpha)^4 \\ &\geq 20c_{201}(17)^4 = 20 \left(\frac{49}{100} \right)^4 > 1. \end{aligned}$$

Hence

$$(3 + \alpha)c(\alpha)^4 \geq 1, \quad \frac{\alpha}{3 + \alpha} \leq \frac{17}{20}, \quad \alpha^4 \leq 1.$$



Estimate 5: Uniform Numerical Verification

Estimate 5. Recall

$$H_n := \frac{1}{4} + \frac{7}{12(n-1)} + \frac{24n(n-1)(n-2)}{(n-4)^4} + \frac{n(n-1)(n-2)}{48} \left(\frac{17}{20}\right)^{(n-4)/2}.$$

The two rational terms decrease for $n \geq 201$. For the last term h_n , the ratio satisfies

$$\frac{h_{n+1}}{h_n} = \frac{n+1}{n-2} \sqrt{\frac{17}{20}} < 1 \quad (n \geq 201).$$

Hence $H_n \leq H_{201}$. Since $\sqrt{17/20} < 461/500$, rational arithmetic gives

$$\begin{aligned} H_{201} &< \frac{1}{4} + \frac{7}{2400} + \frac{24 \cdot 201 \cdot 200 \cdot 199}{197^4} \\ &\quad + \frac{201 \cdot 200 \cdot 199}{48} \left(\frac{17}{20}\right)^{98} \frac{461}{500} < 0.398997 < 0.399 < 1. \end{aligned}$$

Thus $1 \leq R_n(\alpha) \leq H_n < 1$, a contradiction. □

The Case $5 \leq n \leq 200$

Recall the necessary inequality for a counterexample, valid for all $n \geq 5$:

$$1 \leq R_n(\alpha) := \frac{1}{6} + \frac{1}{4(3+\alpha)} + \frac{1}{2N} + \frac{1}{4N(3+\alpha)} + \frac{a^4 n N(n-2)}{4(3+\alpha)} \int_0^1 t^3 Q(t)^{(n-4)/2} dt.$$

It remains to exclude $5 \leq n \leq 200$. For each fixed n , we must prove

$$R_n(\alpha) < 1 \quad \text{whenever} \quad 0 \leq \alpha \leq \min \left\{ 17, \frac{N}{2} \right\}, \quad c(\alpha)^2 \leq a^2.$$

For even $n = 2m + 4$, expand the integral exactly:

$$\int_0^1 t^3 Q(t)^m dt = \sum_{\substack{i,j \geq 0 \\ i+j \leq m}} \frac{m!}{i! j! (m-i-j)!} \frac{(-2c(\alpha))^i a^{2j}}{4+i+2j}.$$

For odd $n = 2m + 5$, one can use the rational upper bound

$$Q(t)^{m+\frac{1}{2}} \leq \frac{1}{2} (Q(t)^m + Q(t)^{m+1}),$$

with a sharper tangent bound for $n = 5$.

Rigorous Verification

After clearing positive denominators, positive Bernstein-basis coefficients provide exact algebraic certificates checked in Lean, not floating-point sampling.

Example: $n = 6$

For $n = 6$, we have $N = 5$ and

$$\int_0^1 t^3 Q(t) dt = \frac{1}{4} - \frac{2c(\alpha)}{5} + \frac{a^2}{6}.$$

Hence direct simplification gives

$$1 - R_6(\alpha) = \frac{P_6(\alpha)}{375(3 + \alpha)^2},$$

where

$$P_6(\alpha) = 1575 - 900\alpha + 2345\alpha^2 - 342\alpha^3 - 24\alpha^4.$$

For $0 \leq \alpha \leq N/2 = 5/2$, we have $342\alpha + 24\alpha^2 \leq 1005$. Therefore

$$\begin{aligned} P_6(\alpha) &= 1575 - 900\alpha + \alpha^2(2345 - 342\alpha - 24\alpha^2) \\ &\geq 1575 - 900\alpha + 1340\alpha^2 > 0, \end{aligned}$$

since the final quadratic has negative discriminant.

Contradiction for $n = 6$

$$1 \leq R_6(\alpha) < 1.$$

Conclusion

The counterexample hypothesis forces

$$\delta < \frac{\alpha}{3 + \alpha}, \quad 1 \leq R_n(\alpha), \quad 0 < \alpha \leq 17.$$

The final estimates show that these conditions are incompatible:

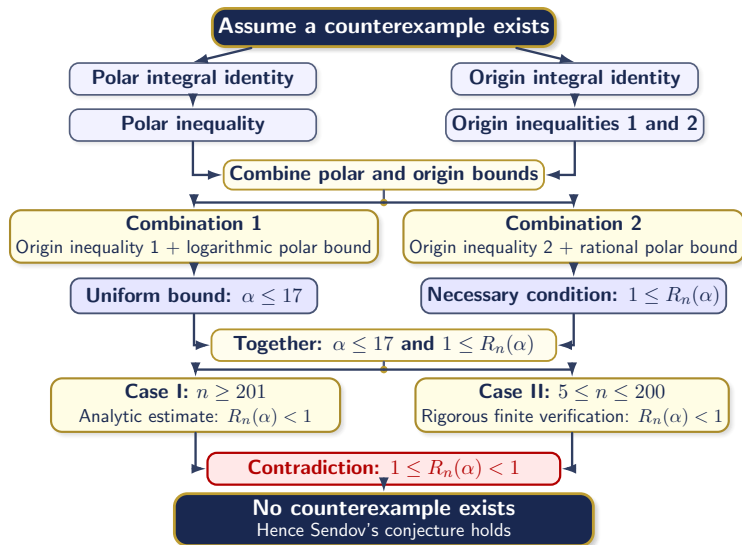
- For $n \geq 201$, an explicit analytic estimate yields $R_n(\alpha) < 0.399$.
- For $5 \leq n \leq 200$, exact algebraic positivity certificates rule out $1 \leq R_n(\alpha)$.
- The low-degree cases and the boundary case $a = 1$ are handled separately.

Final Contradiction

$$1 \leq R_n(\alpha) < 1.$$

Thus the interior case of Sendov's conjecture holds; together with the boundary case this proves the conjecture.

Proof Strategy for Sendov's Conjecture



Borcea's Variance Conjecture

Let P be a polynomial of degree $n \geq 2$ with zeros $z_1, \dots, z_n$. Define

$$\sigma_p(P) := \min_{c \in \mathbb{C}} \left(\frac{1}{n} \sum_{j=1}^n |z_j - c|^p \right)^{1/p}, \quad 1 \leq p < \infty,$$
$$\sigma_\infty(P) := \min_{c \in \mathbb{C}} \max_{1 \leq j \leq n} |z_j - c|.$$

Clearly, by the power mean inequality, $\sigma_p(P)$ is increasing on p .

Conjecture (Borcea)

For every zero a of P , there exists a critical point ζ such that

$$|\zeta - a| \leq \sigma_p(P).$$

The weakest case, corresponding to the limiting exponent $p = \infty$, is equivalent to Sendov's conjecture.

Recent Progress: the case $p \geq 2$ (Zhang¹, 2026)

¹T. Zhang, *Borcea's 2-variance conjecture*, preprint (2026), arXiv:2610.02035.

Schmeisser's Conjecture

Conjecture (Schmeisser)

Let P be a polynomial of degree $n \geq 2$ whose zeros $z_1, \dots, z_n$ lie in the closed unit disk. For every $a \in \operatorname{conv}\{z_1, \dots, z_n\}$, there exists a critical point ζ of P such that $|\zeta - a| \leq 1$.

Recent Progress: For the centroid $G := \frac{1}{n} \sum_{j=1}^n z_j$, Zhang² established the stronger estimate $|\zeta - G| \leq \sqrt{\frac{n-2}{n-1}}$.

²T. Zhang, *A refinement of Pawlowski's result*, Proc. Amer. Math. Soc. **154** (2026), no. 2, 775–782.

Zhang's Generalized Borcea–Schmeisser Conjecture

Recall

$$\sigma_p(P) = \min_{c \in \mathbb{C}} \left(\frac{1}{n} \sum_{j=1}^n |z_j - c|^p \right)^{1/p}.$$

Conjecture (Zhang's Generalized Borcea–Schmeisser Conjecture)

Let P be a polynomial of degree $n \geq 2$. For every $p \geq 1$ and every point $a \in \operatorname{conv}\{z_1, \dots, z_n\}$, there exists a critical point ζ of P such that $|\zeta - a| \leq \sigma_p(P)$.

This conjecture contains both Borcea's variance conjecture and Schmeisser's conjecture as special cases.

Recent Progress: for the centroid $G := \frac{1}{n} \sum_{j=1}^n z_j$, Zhang³ established the stronger estimate $|\zeta - G| \leq \sqrt{\frac{n-2}{n-1}}$.

³T. Zhang, *A refinement of Pawłowski's result*, Proc. Amer. Math. Soc. **154** (2026), no. 2, 775–782.

Tang–Zhang Conjecture

Conjecture (Tang–Zhang)

Let P be a polynomial of degree $n \geq 2$ whose zeros lie in the closed unit disk. Let a be a zero of P , and let $\zeta_1, \dots, \zeta_{n-1}$ be its critical points. For every $\lambda \geq 1$,

$$\sum_{j=1}^{n-1} \frac{1}{|a - \zeta_j|^\lambda} \geq n - 1.$$

The limiting case $\lambda = \infty$ recovers Sendov's conjecture. The case $\lambda = 1$ is the strongest.

Recent Progress: the case $\lambda \geq 2$ (Zhang⁴, 2026)

⁴T. Zhang, *Beyond Sendov's conjecture: the quadratic Tang–Zhang inequality*, preprint (2026), arXiv:2609.19126.

Smale's Mean Value Conjecture

Conjecture (Smale)

Let P be a complex polynomial of degree $n \geq 2$. For every zero a of P , there exists a critical point ζ such that

$$|P(\zeta)| \leq \left(1 - \frac{1}{n}\right) |\zeta - a| |P'(a)|.$$

The constant $1 - \frac{1}{n}$ is attained by

$$P(z) = z^n - z, \quad a = 0.$$

Smale proved the corresponding inequality with the constant 4.

Recent Progress: **Disproved!** A counterexample to the inequality with constant 1, formalized in Lean 4, was reported in September 2026⁵.

⁵T. Adamczewski, *Smale's mean value conjecture ($K = 1$): a Lean 4 disproof*, GitHub repository (2026).

Thank you for your attention!

Questions?