

THE SQUARE LATTICE IS EXTREMAL FOR THE RECTANGULAR LANDAU PROBLEM

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ABSTRACT. We solve the Landau extremal problem restricted to rectangular lattices of fixed covering radius. The square lattice is the unique extremal lattice, and the maximal derivative at the center of the normalized universal covering is

$$\frac{\Gamma(1/4)^2}{2\sqrt{2}\pi^{3/2}}.$$

Consequently, the rectangular Landau constant is the second lemniscate constant. We also prove uniqueness and real-analytic dependence of the relevant accessory parameter.

1. INTRODUCTION

Let $\mathcal{O}(\mathbb{D})$ denote the holomorphic functions on the unit disk $\mathbb{D}$. For a nonconstant $f \in \mathcal{O}(\mathbb{D})$, we define

$$\ell(f) := \sup\{r > 0 : \mathbb{D}(w, r) \subset f(\mathbb{D}) \text{ for some } w \in \mathbb{C}\},$$

where $\mathbb{D}(w, r) = \{z \in \mathbb{C} : |z - w| < r\}$. Landau [Lan29] proved the following theorem.

Landau's theorem. *Let f be holomorphic in the unit disk $\mathbb{D}$ with $f'(0) = 1$. Then there exists an absolute constant $C > 0$, independent of f , such that $\ell(f) \geq C$.*

The largest possible constant C in Landau's theorem is called *Landau's constant*, and we denoted it by L . Equivalently,

$$L = \inf_{\substack{f \in \mathcal{O}(\mathbb{D}) \\ f'(0) \neq 0}} \frac{\ell(f)}{|f'(0)|}, \quad \frac{1}{L} = \sup_{\substack{f \in \mathcal{O}(\mathbb{D}) \\ f'(0) \neq 0}} \frac{|f'(0)|}{\ell(f)}.$$

The problem of determining the exact value of L , posed by Landau [Lan29], is known as the *Landau constant problem* and remains open. Ahlfors's curvature version of Schwarz's lemma [Ahl38] gives the classical lower bound $L \geq 1/2$. Rademacher [Rad43] obtained the upper bound

$$L \leq L_{\text{hex}} := \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} = 0.5432589653\dots,$$

and conjectured that $L = L_{\text{hex}}$, which is called *Landau's constant conjecture*.

Landau's constant conjecture. *Let $\Lambda_{\text{hex}} := \{m + ne^{i\pi/3} : m, n \in \mathbb{Z}\}$ and $w_{\text{hex}} := (1 + e^{i\pi/3})/3$. Let $g_{\text{hex}} : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_{\text{hex}}$ be a universal covering with $g_{\text{hex}}(0) = w_{\text{hex}}$ and $g'_{\text{hex}}(0) > 0$. Then*

$$\sup_{\substack{f \in \mathcal{O}(\mathbb{D}) \\ f'(0) \neq 0}} \frac{|f'(0)|}{\ell(f)} = \frac{g'_{\text{hex}}(0)}{\ell(g_{\text{hex}})}.$$

Equivalently, the Landau's constant $L = L_{\text{hex}}$.

Date: September 12, 2026.

2020 Mathematics Subject Classification. Primary 30F45; Secondary 30C20, 30C25, 65G20.

Key words and phrases. Landau constant; hyperbolic metric; rectangular lattice; Lamé equation; interval arithmetic.

The hexagonal lattice and the conjectured extremal map for Landau's constant

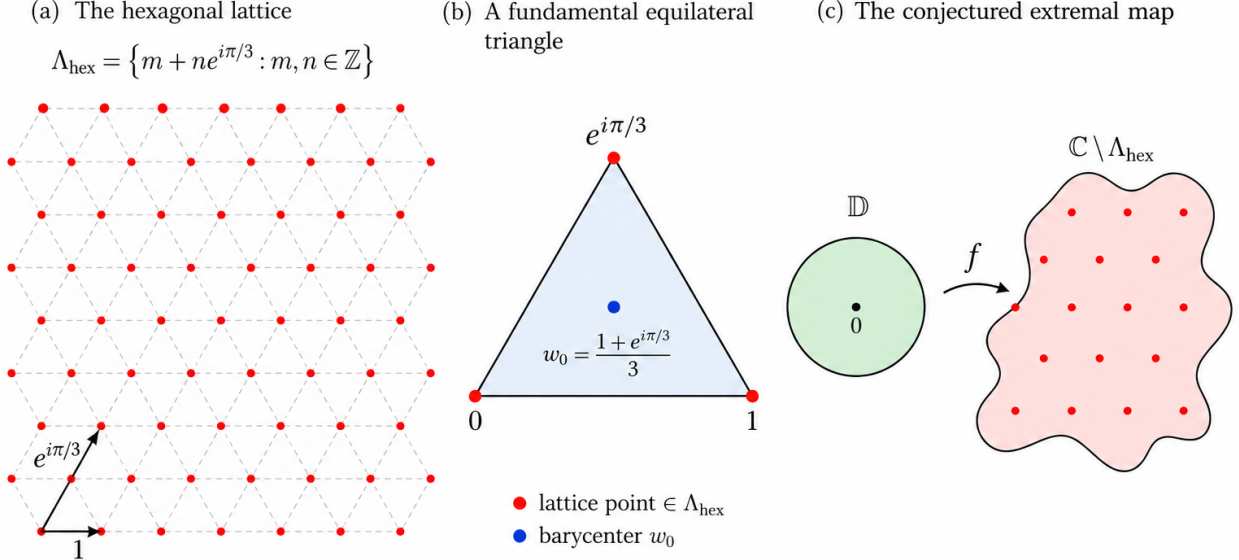


Figure: (a) The hexagonal lattice $\Lambda_{\text{hex}} = \{m + ne^{i\pi/3} : m, n \in \mathbb{Z}\}$. (b) A fundamental equilateral triangle with vertices 0 , 1 , $e^{i\pi/3}$ and barycenter $w_0 = (1 + e^{i\pi/3})/3$. (c) The conjectured extremal function for Landau's constant is, up to natural normalizations, a universal covering map $f : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_{\text{hex}}$ with $f(0) = w_0$.

Yanagihara [Yan95] proved that $L > 1/2 + 10^{-335}$, and Chen and Shiba [CS04] later improved this to $L > 1/2 + 2 \cdot 10^{-8}$. Rettinger [Ret12] showed that Landau's constant is a computable real number. Very recently, Wikström [Wik26] proved the bound $L > 0.51$ by rigorous computer-assisted methods.

A natural variation of Landau's problem is obtained by restricting the target to complements of discrete subsets of the plane. Baernstein [Bae94] formulated several extremal problems of this type, and later studied a closely related extremal problem for heat kernels on flat tori in [Bae97]. Baernstein and Vinson [BV98] proved local minimality results arising in connection with the Bloch and Landau constants. Subsequently, Baernstein, Eremenko, Fryntov, and Solynin [BEFS05] obtained sharp Landau-type covering theorems by studying extremal properties of the hyperbolic metric.

In the same paper [BEFS05], they also considered the rectangular-lattice case and conjectured that the square lattice is extremal among rectangular lattices, with the optimal base point at the center of a fundamental square. They proved the conjecture for sufficiently degenerate rectangles, established the symmetry under interchange of the two side lengths, computed the square case explicitly, and analyzed the limiting behavior in the degenerate regime.

Rectangular Landau extremal problem. For $a, b > 0$, let $\Lambda_{a,b} := a\mathbb{Z} + ib\mathbb{Z}$. For each $a, b > 0$, let $f : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_{a,b}$ be a universal covering. It is conjectured that

$$\sup_{\substack{a,b>0 \\ f:\mathbb{D}\rightarrow\mathbb{C}\setminus\Lambda_{a,b} \text{ universal covering}}} \frac{|f'(0)|}{\ell(f)} = \frac{g'_{\square}(0)}{\ell(g_{\square})},$$

where $\Lambda_{\square} := \mathbb{Z} + i\mathbb{Z}$ and $g_{\square} : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_{\square}$ is the universal covering normalized by $g_{\square}(0) = (1 + i)/2$ and $g'_{\square}(0) > 0$.

Rectangular lattices and the conjectured extremal map for the rectangular-lattice Landau problem

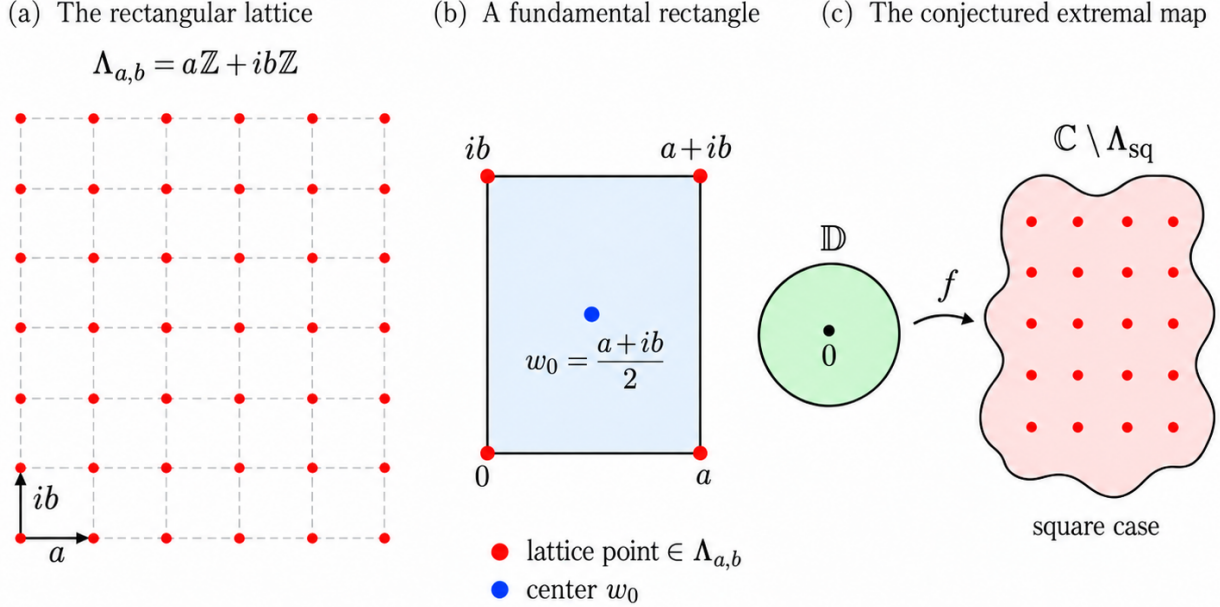


Figure: (a) The rectangular lattice $\Lambda_{a,b} = a\mathbb{Z} + ib\mathbb{Z}$. (b) A fundamental rectangle with vertices $0, a, a + ib, ib$ and center $w_0 = (a + ib)/2$. (c) The conjectured extremal function for the rectangular-lattice Landau problem is, up to natural normalizations, the universal covering map $g_{sq} : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_{sq}$, where $\Lambda_{sq} = \mathbb{Z} + i\mathbb{Z}$, normalized by $g_{sq}(0) = (1 + i)/2$ and $g'_{sq}(0) > 0$.

For a fixed rectangular lattice, the choice of the extremal base point is already known. Sugawa [Sug02] proved that the hyperbolic density of a once-punctured rectangular torus attains its unique minimum at the center of a fundamental rectangle. Hence, once the lattice is fixed, the extremal base point is uniquely determined, and the remaining problem is to compare rectangular lattices of different aspect ratios.

Eremenko [Ere09] subsequently reformulated the problem in a normalization with fixed covering radius. In [Ere11], he reduced the problem to the determination of an accessory parameter in a Lamé equation and described a numerical procedure for computing it. This reduction is closely related to the classical theory of accessory parameters for punctured tori; see, for example, [KRV79].

The square case has also been studied from several related viewpoints. The corresponding extremal value was computed explicitly in [BEFS05]. Faulhuber [Fau21, p. 4 and §7] later identified this value with Gauss' constant and the associated rectangular Landau constant with the second lemniscate constant. Bétermin, Faulhuber, and Steinerberger [BFS21, §2.6] subsequently stated the same square-lattice extremality conjecture.

The extremal number is Gauss' constant G . Equivalently, the rectangular Landau constant is $1/(2G)$, the second lemniscate constant; see Corollary 7.2. In addition, the proof gives two useful by-products. First, the accessory parameter singled out in [Ere11] is unique for every rectangular modulus and depends real-analytically on that modulus; see Corollary 4.4. Second, the certified second-derivative estimate near the square yields the quantitative stability bound in Corollary 6.1.

The proof has an analytic part and a finite computer-assisted part. The analytic part reduces the problem to one real modulus, recalls Eremenko's Lamé representation, and proves a variational identity that yields strict monotonicity of the matching function. The finite verification proves

strict monotonicity on a compact interval and strict concavity near the square. The remaining thin rectangles are handled by an exact odd-sublattice inclusion, so no limiting numerical computation is needed as the aspect ratio degenerates.

The computer-assisted component is intended to be independently reproducible. The ancillary submission consists of the C++ verifier, the complete certificate of 843 closed parameter intervals, a verification log, and a README recording cryptographic hashes. The verifier uses explicit theta-series tail estimates, validated Taylor integration for the differential equations, and outwardly rounded IEEE-754 binary64 interval arithmetic; it calls no external special-function or ODE library. Section 5 states the exact inequalities certified by the program and explains why their verification implies the required mathematical assertions.

The paper is organized as follows. Section 3 introduces the one-variable normalization, its symmetry, and the square value. Section 4 recalls the Lamé reduction and proves uniqueness and analytic dependence of the accessory parameter. Section 5 gives the rigorous numerical verification. Section 6 combines the finite certificate with the exact sublattice argument to prove Theorem 2.1 and a local stability estimate. Section 7 identifies the rectangular Landau constant and records a concavity question whose affirmative answer would remove the computer-assisted step.

Acknowledgments and AI tools disclosure. The author would like to thank Professor Eremenko for suggesting this problem, for his support, and for his helpful comments.

ChatGPT was used for English-language editing, proofreading, grammatical corrections, and as an exploratory tool for discussing possible approaches to selected parts of this paper under the author's mathematical guidance and corrections. The mathematical arguments and proofs in the original manuscript were independently developed, checked, and written by the authors.

2. REDUCTION

We now reduce the rectangular Landau extremal problem to a one-parameter problem. The quotient $|f'(0)|/\ell(f)$ is invariant under dilations of the target. Indeed, if $c > 0$, then cf is a universal covering of $\mathbb{C} \setminus c\Lambda$ and

$$|(cf)'(0)| = c|f'(0)|, \quad \ell(cf) = c\ell(f).$$

Hence

$$\frac{|(cf)'(0)|}{\ell(cf)} = \frac{|f'(0)|}{\ell(f)}.$$

We may therefore fix the scale of the rectangular lattice.

For $\Lambda_{a,b} = a\mathbb{Z} + ib\mathbb{Z}$, the largest Euclidean disk contained in $\mathbb{C} \setminus \Lambda_{a,b}$ is centered at the center of a fundamental rectangle and has radius

$$\ell(f) = \frac{1}{2}\sqrt{a^2 + b^2}.$$

Thus, after a dilation, we may impose the normalization $a^2 + b^2 = 1$, so that the covering radius is fixed at $\ell(f) = 1/2$.

Writing $b = (1 - a^2)^{1/2}$, where $0 < a < 1$, set $\Lambda_a = \{an + ibm : n, m \in \mathbb{Z}\}$, and let $f_a : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_a$ be the universal covering normalized by $f_a(0) = (a + ib)/2$ and $f'_a(0) > 0$. Since $\ell(f_a) = 1/2$, we have

$$\frac{|f'_a(0)|}{\ell(f_a)} = 2f'_a(0).$$

Consequently, maximizing $|f'(0)|/\ell(f)$ over rectangular lattices is equivalent to maximizing $f'_a(0)$ over $0 < a < 1$.

Under the normalization $a^2 + b^2 = 1$, the lattice is square precisely when $a = b$, which is equivalent to

$$a = b = \frac{1}{\sqrt{2}}.$$

Therefore the rectangular Landau extremal problem is equivalent to the assertion that $a \mapsto f'_a(0)$ attains its maximum at $a = 1/\sqrt{2}$.

Our main theorem proves this assertion and determines the maximal value.

Theorem 2.1. *One has*

$$\max_{0 < a < 1} f'_a(0) = \frac{\Gamma(1/4)^2}{2\sqrt{2}\pi^{3/2}} = 0.834626841674073\dots,$$

and the unique maximizing parameter is $a = 1/\sqrt{2}$.

3. NORMALIZATION, SYMMETRY, AND THE SQUARE VALUE

It is convenient to move the center of a fundamental rectangle to the origin and to divide by the vertical period. For $t > 0$, define

$$\Omega_t = \mathbb{C} \setminus \left\{ t \left(n + \frac{1}{2} \right) + i \left(m + \frac{1}{2} \right) : n, m \in \mathbb{Z} \right\}.$$

Let $g_t : \mathbb{D} \rightarrow \Omega_t$ be the universal covering normalized by $g_t(0) = 0$ and $g'_t(0) > 0$, and write

$$r(t) := g'_t(0).$$

We shall use the following symmetry in exactly the normalization in which it appears in the earlier literature. We state it explicitly so that the subsequent specialization is transparent.

Lemma 3.1 ([BEFS05, p. 127, §8]). *For $k > 0$, let*

$$\Lambda_k = 2\pi k\mathbb{Z} + 2\pi i\mathbb{Z},$$

and let

$$F_k : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda_k$$

be the universal covering normalized by

$$F_k(0) = \pi(k + i), \quad F'_k(0) > 0.$$

If

$$a_{\text{BEFS}}(k) := F'_k(0),$$

then

$$a_{\text{BEFS}}(k^{-1}) = k^{-1}a_{\text{BEFS}}(k), \quad k > 0. \tag{3.1}$$

The next lemma is the form of this relation needed in the present fixed-covering-radius normalization.

Lemma 3.2. *If $t = a/b$, then*

$$f'_a(0) = R(t) := \frac{r(t)}{\sqrt{1+t^2}}. \tag{3.2}$$

Moreover,

$$r(1/t) = \frac{r(t)}{t}, \quad R(1/t) = R(t). \tag{3.3}$$

Proof. Translation by $-(a + ib)/2$ followed by division by b sends the puncture set of $\mathbb{C} \setminus \Lambda_a$ to the puncture set of Ω_t , where $t = a/b$. Hence

$$g_t(z) = \frac{f_a(z) - (a + ib)/2}{b},$$

and therefore $r(t) = f'_a(0)/b$. Since $a^2 + b^2 = 1$ and $t = a/b$, we have $b = (1 + t^2)^{-1/2}$, which gives (3.2).

To obtain the symmetry, apply Lemma 3.1 with $k = t$. After translating the center to 0 and dividing the target by 2π , one has

$$r(t) = \frac{a_{\text{BEFS}}(t)}{2\pi}.$$

Thus (3.1) gives

$$r(1/t) = \frac{a_{\text{BEFS}}(1/t)}{2\pi} = \frac{a_{\text{BEFS}}(t)}{2\pi t} = \frac{r(t)}{t}.$$

Substitution into (3.2) yields $R(1/t) = R(t)$. □

Normalization check. The phase normalization in the preceding argument is automatic because Lemma 3.1 is stated for coverings with positive real derivative. Equivalently, a direct proof in the present notation uses

$$h(\zeta) = \frac{i}{t} g_t(-i\zeta),$$

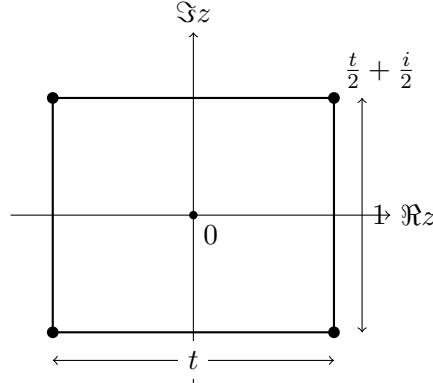
for which $h : \mathbb{D} \rightarrow \Omega_{1/t}$, $h(0) = 0$, and $h'(0) = r(t)/t > 0$.

We shall work with the square

$$Q(t) := R(t)^2 = \frac{r(t)^2}{1+t^2}.$$

By Lemma 3.2, $Q(1/t) = Q(t)$. Once differentiability near $t = 1$ is established in Section 4, this symmetry gives $Q'(1) = 0$.

The following diagram shows the centered fundamental rectangle for Ω_t .



The centered fundamental rectangle for Ω_t ; its four corners are punctures.

The square value is also available explicitly in the same source normalization.

Lemma 3.3 ([BEFS05, p. 129, §9]). *With $a_{\text{BEFS}}(k)$ as in Lemma 3.1,*

$$a_{\text{BEFS}}(1) = B(1/4, 1/4) = \frac{\Gamma(1/4)^2}{\sqrt{\pi}}. \tag{3.4}$$

Proposition 3.4. *One has*

$$r(1) = \frac{\Gamma(1/4)^2}{2\pi^{3/2}}$$

and therefore

$$R(1) = \frac{\Gamma(1/4)^2}{2\sqrt{2}\pi^{3/2}} =: G.$$

The number G is Gauss' constant.

Proof. Apply Lemma 3.3 at $k = 1$. Translating the square center to the origin and dividing the target by 2π gives Ω_1 , so

$$r(1) = \frac{a_{\text{BEFS}}(1)}{2\pi} = \frac{\Gamma(1/4)^2}{2\pi^{3/2}}.$$

Now (3.2) with $t = 1$ gives the displayed value of $R(1)$. Its identification with Gauss' constant is recorded in [Fau21, p. 4]. $\square$

For later comparison we record a convenient elementary lower bound. The identity

$$G = \theta_4(e^{-\pi})^2, \quad \theta_4(q) = 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}, \quad (3.5)$$

is given in [Fau21, p. 4]. Since $e^{-\pi} < 1/23$ (for example, $\pi > 3.14$ and the first ten positive terms of the exponential series give $e^{3.14} > 23$), and the alternating series in (3.5) gives $\theta_4(q) > 1 - 2q$ for $0 < q < 1$, we obtain

$$Q(1) = G^2 > \left(\frac{21}{23}\right)^4 > \frac{69}{100}. \quad (3.6)$$

4. THE LAMÉ EQUATION AND THE ACCESSORY PARAMETER

We next isolate the precise part of Eremenko's Lamé construction that will be used; the notation and geometric construction are taken from [Ere11, pp. 5–10]. The statement below is written self-containedly in the notation of the source, with the two circle radii denoted by ρ, ρ_1 to avoid conflict with our function $r(t)$.

Lemma 4.1 ([Ere11, p. 7, Proposition]). *Let $\omega > 0$ and $\omega'/i > 0$, and let $\wp$ be the Weierstrass function with periods 2ω and $2\omega'$. Put*

$$e_2 = \wp(\omega + \omega'), \quad P(\zeta) = \frac{1}{4}(\wp(\zeta + \omega + \omega') - e_2).$$

For real λ , let c, s solve

$$w'' + Pw = \lambda w, \quad c(0) = s'(0) = 1, \quad c'(0) = s(0) = 0.$$

Let λ_- be the largest Neumann eigenvalue on $[0, \omega]$, and let λ_+ be the smallest Neumann eigenvalue of the equation obtained on the imaginary axis $\zeta = iy$, $0 \leq y \leq \omega'/i$. Define

$$\begin{aligned} m &= \frac{1}{2} \left(\frac{s(\omega)}{c(\omega)} + \frac{s'(\omega)}{c'(\omega)} \right), & \rho &= \frac{1}{2} \left(\frac{s(\omega)}{c(\omega)} - \frac{s'(\omega)}{c'(\omega)} \right), \\ m_1 &= -\frac{i}{2} \left(\frac{s(\omega')}{c(\omega')} + \frac{s'(\omega')}{c'(\omega')} \right), & \rho_1 &= -\frac{i}{2} \left(\frac{s(\omega')}{c(\omega')} - \frac{s'(\omega')}{c'(\omega')} \right). \end{aligned}$$

Then there is a unique $\lambda \in [\lambda_-, \lambda_+]$ satisfying

$$\frac{m_1^2}{\sqrt{m^2 + m_1^2}} = \rho_1. \quad (4.1)$$

For this value of λ , $F = s/c$ is the inverse developing map of the normalized universal covering, $F(0) = 0$, $F'(0) = 1$, and the covering derivative is

$$A = \frac{mm_1}{\sqrt{m^2 + m_1^2}}. \quad (4.2)$$

Moreover, the images of the two outer sides of the quarter rectangle are tangent circles centered on the real and imaginary axes, with the above center parameters and signed radii.

Uniqueness itself is already part of the cited proposition. What we prove below is the stronger differential fact that the matching function has strictly negative λ -derivative. This is the property needed for branch identification in the validated continuation.

For our lattice, take in Lemma 4.1

$$\omega = \frac{t}{2}, \quad \omega' = \frac{i}{2}.$$

Thus

$$P_t(z) = \frac{1}{4} \left(\wp_t \left(z + \frac{t+i}{2} \right) - \wp_t \left(\frac{t+i}{2} \right) \right), \quad (4.3)$$

where $\wp_t$ has periods t and i . If Φ_t denotes the local inverse of g_t at 0, then $\Phi_t'(0) = 1/r(t)$. Hence

$$F_t := r(t)\Phi_t$$

satisfies $F_t(0) = 0$ and $F_t'(0) = 1$. Since a nonzero affine postcomposition does not change the Schwarzian derivative, Lemma 4.1 gives

$$\{F_t, z\} = 2(P_t(z) - \lambda),$$

and therefore $F_t = s/c$, where

$$y'' = (\lambda - P_t)y, \quad c(0) = s'(0) = 1, \quad c'(0) = s(0) = 0. \quad (4.4)$$

For the validated calculation we put both symmetry axes on $[0, 1]$. Define C, S, D, U by

$$\begin{aligned} C'' &= V_x C, & S'' &= V_x S, & V_x(x) &= \frac{t^2}{4} (\lambda - P_t(tx/2)), \\ D'' &= V_y D, & U'' &= V_y U, & V_y(x) &= \frac{1}{4} (P_t(ix/2) - \lambda), \end{aligned}$$

with

$$\begin{aligned} C(0) &= D(0) = 1, & C'(0) &= D'(0) = 0, \\ S(0) &= U(0) = 0, & S'(0) &= U'(0) = 1. \end{aligned}$$

Then

$$c(tx/2) = C(x), \quad s(tx/2) = \frac{t}{2} S(x),$$

and

$$c(ix/2) = D(x), \quad s(ix/2) = \frac{i}{2} U(x).$$

From now on, $C, C', S, S', D, D', U, U'$ without an argument are evaluated at $x = 1$.

Applying the definitions in Lemma 4.1 gives

$$\begin{aligned} m &= \frac{t}{4} \left(\frac{S}{C} + \frac{S'}{C'} \right), & \rho &= \frac{t}{4} \left(\frac{S}{C} - \frac{S'}{C'} \right), \\ m_1 &= \frac{1}{4} \left(\frac{U}{D} + \frac{U'}{D'} \right), & \rho_1 &= \frac{1}{4} \left(\frac{U}{D} - \frac{U'}{D'} \right). \end{aligned}$$

Consequently, the powers of the origin with respect to the two circles are

$$\mathcal{R}_x(t, \lambda) = m^2 - \rho^2 = \frac{t^2 SS'}{4CC'}, \quad \mathcal{R}_y(t, \lambda) = m_1^2 - \rho_1^2 = \frac{UU'}{4DD'}. \quad (4.5)$$

At the matching value, (4.1) and (4.2) give

$$\mathcal{R}_x(t, \lambda) = \mathcal{R}_y(t, \lambda) = r(t)^2.$$

Thus, with

$$\mathcal{F}(t, \lambda) = t^2 SS' DD' - UU' CC', \quad \mathcal{Q}(t, \lambda) = \frac{t^2 SS'}{4(1+t^2)CC'}, \quad (4.6)$$

the accessory parameter $\lambda(t)$ satisfies

$$\mathcal{F}(t, \lambda(t)) = 0, \quad Q(t) = Q(t, \lambda(t)). \quad (4.7)$$

The following elementary first-variation identity will be used twice. We include the short Wronskian proof because it is more transparent here than invoking general Sturm–Liouville perturbation theory.

Lemma 4.2. *Let $V, W \in C([0, L]; \mathbb{R})$. For $|\varepsilon|$ small, let $c_\varepsilon, s_\varepsilon$ solve*

$$y'' = (V + \varepsilon W)y$$

with

$$c_\varepsilon(0) = s'_\varepsilon(0) = 1, \quad c'_\varepsilon(0) = s_\varepsilon(0) = 0.$$

Assume $c_0(L)c'_0(L) \neq 0$, and set

$$p_\varepsilon = \frac{s_\varepsilon(L)}{c_\varepsilon(L)}, \quad q_\varepsilon = \frac{s'_\varepsilon(L)}{c'_\varepsilon(L)}.$$

With a dot denoting $d/d\varepsilon$ at $\varepsilon = 0$ and $c = c_0$, $s = s_0$, one has

$$\dot{p} = - \int_0^L (s - pc)^2 W \, dx, \quad \dot{q} = - \int_0^L (s - qc)^2 W \, dx. \quad (4.8)$$

Proof. Let $B_0 y = y(L)$ and $B_1 y = y'(L)$, and put

$$m_{j,\varepsilon} = \frac{B_j s_\varepsilon}{B_j c_\varepsilon}, \quad y_{j,\varepsilon} = s_\varepsilon - m_{j,\varepsilon} c_\varepsilon, \quad j = 0, 1.$$

Then $B_j y_{j,\varepsilon} = 0$. At $\varepsilon = 0$, write $y_j = y_{j,0}$. Differentiating the differential equation gives

$$\dot{y}_j'' = V \dot{y}_j + W y_j,$$

so

$$(y_j \dot{y}_j' - y_j' \dot{y}_j)' = W y_j^2.$$

The boundary term at L vanishes: for $j = 0$ because $y_0(L) = \dot{y}_0(L) = 0$, and for $j = 1$ because $y_1'(L) = \dot{y}_1'(L) = 0$. At 0,

$$y_j(0) = -m_j, \quad y_j'(0) = 1, \quad \dot{y}_j(0) = -\dot{m}_j, \quad \dot{y}_j'(0) = 0,$$

so the Wronskian term equals $\dot{m}_j$. Integrating from 0 to L yields

$$\dot{m}_j = - \int_0^L (s - m_j c)^2 W \, dx.$$

Taking $j = 0$ and $j = 1$ gives (4.8). □

Let $\lambda_-(t), \lambda_+(t)$ be the two principal Neumann thresholds obtained from Lemma 4.1 after the rescaling above, and put

$$I_t = (\lambda_-(t), \lambda_+(t)).$$

The endpoint degenerations in Eremenko's proof show that the geometric matching parameter lies in I_t ; see [Ere11, pp. 9–10]. We now establish the strict monotonicity that will be used later.

Proposition 4.3. *For every $t > 0$,*

$$H_t(\lambda) := \mathcal{R}_x(t, \lambda) - \mathcal{R}_y(t, \lambda)$$

satisfies

$$\partial_\lambda H_t(\lambda) < 0, \quad \lambda \in I_t.$$

Hence the zero furnished by Lemma 4.1 is the unique zero of H_t in I_t , and therefore the unique zero of $\mathcal{F}(t, \cdot)$ there.

Proof. Fix $t > 0$ and $\lambda \in I_t$. By Sturm comparison, the even solutions C, D are positive on $[0, 1]$. If v_- and v_+ denote positive principal Neumann eigenfunctions at $\lambda_-(t)$ and $\lambda_+(t)$, respectively, the Wronskian identity gives

$$C'(1)v_-(1) = \frac{t^2}{4}(\lambda - \lambda_-(t)) \int_0^1 C(x)v_-(x) dx > 0,$$

and

$$D'(1)v_+(1) = \frac{1}{4}(\lambda_+(t) - \lambda) \int_0^1 D(x)v_+(x) dx > 0.$$

Thus all denominators below are positive.

Set

$$p_x = \frac{S}{C}, \quad q_x = \frac{S'}{C'}, \quad p_y = \frac{U}{D}, \quad q_y = \frac{U'}{D'}.$$

The Wronskian identities $CS' - C'S = 1$ and $DU' - D'U = 1$ give

$$p_x = \int_0^1 C(x)^{-2} dx > 0, \quad q_x = p_x + \frac{1}{CC'} > 0,$$

and analogously $p_y, q_y > 0$.

On the horizontal axis, $\partial_\lambda V_x = t^2/4$. Lemma 4.2 therefore gives

$$\partial_\lambda p_x = -\frac{t^2}{4} \int_0^1 (S - p_x C)^2 dx < 0, \quad \partial_\lambda q_x = -\frac{t^2}{4} \int_0^1 (S - q_x C)^2 dx < 0.$$

Since $\mathcal{R}_x = (t^2/4)p_x q_x$, it follows that $\partial_\lambda \mathcal{R}_x < 0$.

On the vertical axis, $\partial_\lambda V_y = -1/4$, so the same lemma gives

$$\partial_\lambda p_y = \frac{1}{4} \int_0^1 (U - p_y D)^2 dx > 0, \quad \partial_\lambda q_y = \frac{1}{4} \int_0^1 (U - q_y D)^2 dx > 0.$$

Since $\mathcal{R}_y = (1/4)p_y q_y$, we have $\partial_\lambda \mathcal{R}_y > 0$. Therefore

$$\partial_\lambda H_t = \partial_\lambda \mathcal{R}_x - \partial_\lambda \mathcal{R}_y < 0.$$

Finally,

$$\mathcal{F}(t, \lambda) = 4CC'DD' H_t(\lambda),$$

and $CC'DD' > 0$ on I_t , so H_t and $\mathcal{F}(t, \cdot)$ have the same zeros there. $\square$

The strict derivative, rather than uniqueness alone, gives the required nondegeneracy.

Corollary 4.4. *The accessory parameter $\lambda(t)$ is uniquely defined and real analytic for $t > 0$. Moreover,*

$$\mathcal{F}_\lambda(t, \lambda(t)) < 0.$$

Proof. At the zero of H_t ,

$$\mathcal{F}_\lambda = 4CC'DD' \partial_\lambda H_t < 0$$

by Proposition 4.3. The coefficients of the two initial-value problems depend real-analytically on (t, λ) for $t > 0$. The implicit function theorem therefore gives a real-analytic branch locally at every t , and uniqueness makes these local branches agree. $\square$

We shall also need the exact center of the branch at the square.

Lemma 4.5. *One has*

$$\lambda(1) = 0.$$

Proof. For the square lattice, rotation by i and periodicity give

$$P_1(iz) = -P_1(z).$$

Indeed, for the square Weierstrass function $\wp_1(iw) = -\wp_1(w)$, and the half-period shift in (4.3) is preserved modulo the period 1. Hence at $t = 1$ and $\lambda = 0$,

$$V_x(x) = V_y(x) = -\frac{1}{4}P_1(x/2).$$

By uniqueness of the initial-value problems, $C = D$ and $S = U$, so (4.5) gives $H_1(0) = 0$. Proposition 4.3 then yields $\lambda(1) = 0$. $\square$

Proposition 4.3 is the analytic input that makes the finite verification rigorous. In particular, once a certificate box verifies positivity of $C, D, C'(1), D'(1)$ and encloses a zero of $\mathcal{F}$, the strict inequality $\mathcal{F}_\lambda < 0$ identifies that zero with the geometric branch.

Differentiating (4.7) gives

$$Q'(t) = Q_t - Q_\lambda \frac{\mathcal{F}_t}{\mathcal{F}_\lambda} = \frac{Q_t \mathcal{F}_\lambda - Q_\lambda \mathcal{F}_t}{\mathcal{F}_\lambda}. \quad (4.9)$$

If $p = \lambda' = -\mathcal{F}_t/\mathcal{F}_\lambda$, then

$$\lambda'' = -\frac{\mathcal{F}_{tt} + 2\mathcal{F}_{t\lambda}p + \mathcal{F}_{\lambda\lambda}p^2}{\mathcal{F}_\lambda}, \quad Q'' = Q_{tt} + 2Q_{t\lambda}p + Q_{\lambda\lambda}p^2 + Q_\lambda\lambda''. \quad (4.10)$$

These identities are evaluated with interval arithmetic in Section 5.

5. RIGOROUS NUMERICAL VERIFICATION

We now describe the finite verification used in the proof. The purpose of this section is not to present a floating-point experiment, but to specify a sequence of interval inclusions whose successful execution is a mathematical proof. For general background on interval arithmetic and automatic result verification, see Moore, Kearfott, and Cloud [MKC09, pp. 1–50]. The present verifier is self-contained and does not rely on an interval library.

Set

$$c_0 = \frac{7}{64}, \quad \delta = 2^{-16}.$$

The exact output needed for the global argument is summarized next. Proposition 5.1 is part of the proof only together with the ancillary verifier and certificate described below.

Proposition 5.1. *The following inequalities hold:*

$$Q'(t) > 0 \quad \text{for } c_0 \leq t \leq 1 - \delta, \quad (5.1)$$

$$Q''(t) < -\frac{1}{10} \quad \text{for } 1 - \delta \leq t \leq 1 + \delta, \quad (5.2)$$

$$r(21/64)^2 < \frac{2}{3}. \quad (5.3)$$

The rest of this section proves Proposition 5.1 from the supplied verifier and certificate.

5.1. Theta-function representation. Put $q = e^{-\pi/t}$ and use the standard conventions

$$\begin{aligned}\theta_1(v, q) &= 2 \sum_{j \geq 0} (-1)^j q^{(j+1/2)^2} \sin((2j+1)v), \\ \theta_3(v, q) &= 1 + 2 \sum_{j \geq 1} q^{j^2} \cos(2jv), \\ \theta_2(q) &= 2 \sum_{j \geq 0} q^{(j+1/2)^2}, \\ \theta_4(q) &= 1 + 2 \sum_{j \geq 1} (-1)^j q^{j^2}.\end{aligned}$$

The Weierstrass–theta identity used in [Ere11, p. 10, §5], applied to (4.3), gives

$$P_t(z) = -\frac{\pi^2 \theta_2(q)^2 \theta_4(q)^2}{4t^2} \left(\frac{\theta_1(\pi z/t, q)}{\theta_3(\pi z/t, q)} \right)^2. \quad (5.4)$$

Substitution of (5.4) into the two fixed-interval ODEs gives the coefficient functions used by the program.

The verifier keeps the terms $j = 0, \dots, 8$ for θ_1, θ_2 and $j = 1, \dots, 8$ for θ_3, θ_4 . On the full verification range

$$0.099 \leq t \leq 1.001, \quad 0 \leq x \leq 1,$$

the omitted terms and their first two t -derivatives are exponentially small. For the spatial Taylor coefficients of order at most 22, a uniform majorant is

$$2 \sum_{j=9}^{\infty} 10^8 (j+1)^6 \frac{[41(j+1)]^{22}}{22!} \exp\left(-\frac{3(j^2-j)}{1.001}\right) < 10^{-35}. \quad (5.5)$$

Here is the elementary bound behind (5.5). On the imaginary axis, the hyperbolic factors are combined with the powers of q before interval evaluation. Since $t \leq 1.001$, $x \leq 1$, and $\pi > 3$, every omitted term with $j \geq 9$ is bounded by a constant multiple of

$$\exp\left(-\frac{3(j^2-j)}{1.001}\right).$$

Taking at most two t -derivatives contributes only powers of t^{-1} and a polynomial in j of degree at most four; on $t \geq 0.099$ these are dominated by $10^8(j+1)^6$. A spatial derivative of order at most 22 contributes at most $[41(j+1)]^{22}$, and division by $22!$ gives the displayed majorant. For $j = 9$, the resulting summand is below $2 \cdot 10^{-42}$, while the ratio of successive summands is below $1/2$. Hence the full tail is $< 10^{-35}$. The theta-constant tails are smaller; the verifier adds symmetric radii 10^{-90} and 10^{-80} to the corresponding jet coefficients of θ_2 and θ_4 .

The imaginary-axis expressions are evaluated after combining the hyperbolic sine and cosine factors with the powers of q . Thus no cancellation of exponentially large numbers occurs.

5.2. Elementary functions and outward rounding. All interval operations are performed in IEEE-754 binary64 arithmetic. After each elementary arithmetic operation the lower endpoint is moved outward by one call to `nextafter` toward $-\infty$ and the upper endpoint is moved outward by one call toward $+\infty$. Since IEEE-754 binary64 addition, subtraction, multiplication, and division are correctly rounded in the required round-to-nearest mode, this one-ulp enlargement contains the exact real result. The verifier checks at run time that binary64 is in use, that the rounding mode is round-to-nearest, and that gradual underflow is available.

The constant π is enclosed using Machin’s identity

$$\pi = 16 \arctan(1/5) - 4 \arctan(1/239),$$

with 25 terms of the alternating arctangent series and the first omitted term as remainder. Exponentials are reduced to $|v| \leq 1/16$, evaluated by a degree-16 Taylor polynomial with the remainder bounded by $3|v|^{17}/17!$, and then reconstructed by repeated squaring. Sine and cosine are reduced to $|v| \leq 2$ and evaluated through degrees 41 and 40, with an added symmetric error 10^{-35} . These elementary bounds are far smaller than the interval widths that occur in the final certificate.

5.3. Validated ODE integration. For an equation $y'' = Vy$, write at a step origin

$$V(x_0 + h) = \sum_{n \geq 0} v_n h^n, \quad y(x_0 + h) = \sum_{n \geq 0} a_n h^n.$$

The coefficients satisfy the exact recurrence

$$a_{n+2} = \frac{\sum_{k=0}^n v_k a_{n-k}}{(n+1)(n+2)}. \quad (5.6)$$

The program applies (5.6) simultaneously to the solution and to its first and second derivatives with respect to t and λ .

The following elementary enclosure criterion explains the validation step used by the program.

Lemma 5.2. *Let $\mathcal{Y} \subset \mathbb{R}^d$ be a closed convex box, let $Y_0 \in \mathcal{Y}$, and let $G : [0, h] \times \mathcal{Y} \rightarrow \mathbb{R}^d$ be continuous and locally Lipschitz in its second variable. If*

$$Y_0 + [0, h]G([0, h] \times \mathcal{Y}) \subseteq \mathcal{Y}, \quad (5.7)$$

then the solution of $Y' = G(x, Y)$, $Y(0) = Y_0$, exists on $[0, h]$ and remains in $\mathcal{Y}$.

Proof. Let X be the closed convex set of continuous maps $Y : [0, h] \rightarrow \mathcal{Y}$ and define

$$(\mathcal{P}Y)(x) = Y_0 + \int_0^x G(s, Y(s)) ds.$$

Condition (5.7) gives $\mathcal{P}(X) \subseteq X$. Its image is uniformly bounded and equicontinuous, so $\mathcal{P}$ is compact by Arzelà–Ascoli; it is continuous as well. Schauder’s theorem gives a fixed point, hence a solution on $[0, h]$. Local Lipschitz continuity gives uniqueness, so this fixed point is the exact solution. $\square$

Before each Taylor step the verifier constructs such a box $\mathcal{Y}$ and checks (5.7). It then uses (5.6) through order 14 and, on the same box, encloses the normalized derivatives of orders 15 and 16. Taylor’s theorem with interval remainder bounds the discarded terms for y and y' . If (5.7) fails, the verifier aborts; no unvalidated numerical integrator is used.

The interval $[0, 1]$ is divided into 16, 32, or 64 equal steps according as the parameter center satisfies $t \geq 1/2$, $1/4 \leq t < 1/2$, or $t < 1/4$. All step sizes are dyadic.

5.4. Parameter jets and root enclosure. For a scalar quantity $A(t, \lambda)$ the program stores the six-component jet

$$J(A) = \left(A, \frac{A_t}{100}, \frac{A_\lambda}{100}, \frac{A_{tt}}{20000}, \frac{A_{t\lambda}}{10000}, \frac{A_{\lambda\lambda}}{20000} \right). \quad (5.8)$$

The multiplication, reciprocal, and exponential rules in the source file are exactly the algebraic differentiation rules for (5.8); the jet is not a quadratic approximation to the parameter dependence.

For a certificate interval $T = [t_-, t_+]$, let t_0 be its midpoint, let λ_0 be the certificate center, and set $L = 5(t_+ - t_-)/2$. On the box

$$T \times [\lambda_0 - L, \lambda_0 + L]$$

the program verifies $\mathcal{F}_\lambda < 0$ and the strict inequality

$$L \inf(-\mathcal{F}_\lambda) > |\mathcal{F}(t_0, \lambda_0)| + \frac{t_+ - t_-}{2} \sup |\mathcal{F}_t|. \quad (5.9)$$

The mean value theorem then gives opposite signs for $\mathcal{F}(t, \lambda_0 - L)$ and $\mathcal{F}(t, \lambda_0 + L)$ for every $t \in T$. Thus each $t \in T$ has a unique root in the box. The ODE enclosures also verify positivity of the even

solutions and positivity of all endpoint quantities needed in Proposition 4.3; hence this enclosed root is the accessory parameter of the universal cover.

To prove (5.1), the program evaluates the numerator

$$N = Q_t \mathcal{F}_\lambda - Q_\lambda \mathcal{F}_t$$

of (4.9). On every certificate box it proves

$$N < 0, \quad \mathcal{F}_\lambda < 0,$$

and hence $Q' > 0$.

For (5.2), put

$$T_\square = [1 - \delta, 1 + \delta].$$

Lemma 4.5 gives the exact root center $\lambda(1) = 0$. The verifier applies the same root-enclosure test (5.9) to

$$\mathcal{B}_\square = T_\square \times [-5\delta, 5\delta],$$

with $t_0 = 1$, $\lambda_0 = 0$, and $L = 5\delta$. It first verifies $\mathcal{F}_\lambda < 0$ and (5.9) on $\mathcal{B}_\square$; hence

$$\lambda(t) \in [-5\delta, 5\delta] \quad (t \in T_\square).$$

Only after this branch enclosure is established does the program evaluate (4.10) on $\mathcal{B}_\square$. The resulting enclosure is

$$-0.24433870323634485 \leq Q''(t) \leq -0.1083878345444004,$$

which implies (5.2).

Finally, a box centered at $t = 21/64$ gives

$$0.66582085620831488 \leq r(21/64)^2 \leq 0.66590457013704418,$$

which implies (5.3).

5.5. Certificate coverage and reproducibility. The certificate contains 843 closed intervals. The verifier checks mechanically that the left endpoint of the first interval is $7/64$, that consecutive endpoints agree exactly as binary64 numbers, and that the right endpoint of the last interval is $1 - 2^{-16}$. Hence there are no gaps. Over all 843 boxes, the largest upper bound for the scaled derivative numerator is

$$-8.1679313268132783 \times 10^{-12},$$

and the smallest lower bound for the root-bracketing margin in (5.9) is

$$1.0356172945565005 \times 10^{-7}.$$

Thus every required inequality is strict with an explicit nonzero margin.

The ancillary README records SHA-256 hashes for the verifier, certificate, and verification log, so that the exact files used for the proof can be identified.

On an x86-64 system with GCC and IEEE-754 binary64 arithmetic, the command

```
g++ -O3 -fno-fast-math -ffp-contract=off -fopenmp -std=c++17 verify.cpp -o verify
followed by ./verify certificate.csv reproduces the enclosures above. The program itself
checks the remaining run-time floating-point assumptions before validating any certificate inter-
val.
```

This completes the mathematical reduction of Proposition 5.1 to the finite ancillary certificate.

6. PROOF OF THE GLOBAL EXTREMAL THEOREM

We can now prove Theorem 2.1 with very little computation-specific input.

Proof. By Corollary 4.4, the function Q is real analytic near $t = 1$. The symmetry $Q(1/t) = Q(t)$ from Lemma 3.2 therefore implies $Q'(1) = 0$. Proposition 5.1 and (5.2) give, for $1 - \delta \leq t < 1$,

$$Q'(t) = - \int_t^1 Q''(u) du > 0.$$

Together with (5.1), this proves

$$Q'(t) > 0 \quad (c_0 \leq t < 1). \quad (6.1)$$

Thus $R(t) = \sqrt{Q(t)}$ is strictly increasing on $[c_0, 1]$. Since $r(t) = R(t)\sqrt{1+t^2}$, the function r is also strictly increasing there.

If n is a positive odd integer, then

$$nt \left(\mathbb{Z} + \frac{1}{2} \right) + i \left(\mathbb{Z} + \frac{1}{2} \right) \subseteq t \left(\mathbb{Z} + \frac{1}{2} \right) + i \left(\mathbb{Z} + \frac{1}{2} \right),$$

so $\Omega_t \subseteq \Omega_{nt}$. Let $\iota : \Omega_t \hookrightarrow \Omega_{nt}$ be the inclusion. Since $\mathbb{D}$ is simply connected, $\iota \circ g_t$ lifts through g_{nt} to a holomorphic map $\phi : \mathbb{D} \rightarrow \mathbb{D}$ with $\phi(0) = 0$ and

$$\iota \circ g_t = g_{nt} \circ \phi.$$

Differentiating at 0 and applying Schwarz's lemma gives

$$r(t) = r(nt)|\phi'(0)| \leq r(nt). \quad (6.2)$$

Choose the smallest positive odd integer n for which $nt \geq c_0$. Since $(n-2)t < c_0$,

$$c_0 \leq nt < c_0 + 2t < 3c_0 = \frac{21}{64}.$$

Using the strict increase of r following (6.1), together with (6.2) and (5.3), we obtain

$$Q(t) = \frac{r(t)^2}{1+t^2} < r(t)^2 \leq r(nt)^2 < r(21/64)^2 < \frac{2}{3}.$$

On the other hand, (3.6) gives $Q(1) > 69/100 > 2/3$. Hence $Q(t) < Q(1)$ for every $0 < t < c_0$. Combining this with strict increase on $[c_0, 1)$ yields

$$Q(t) < Q(1) \quad (0 < t < 1).$$

The symmetry $Q(1/t) = Q(t)$ gives the same strict inequality for $t > 1$. Therefore $t = 1$ is the unique global maximizer of R .

Returning to a by $t = a/b$ and $a^2 + b^2 = 1$, the equation $t = 1$ is equivalent to $a = b = 1/\sqrt{2}$. Proposition 3.4 supplies the value of the maximum. $\square$

Corollary 6.1. *Let $\delta = 2^{-16}$. If $|t - 1| \leq \delta$, then*

$$Q(1) - Q(t) \geq \frac{(t-1)^2}{20}.$$

Proof. By symmetry, $Q'(1) = 0$, and Proposition 5.1 gives $Q''(u) \leq -1/10$ on $[1-\delta, 1+\delta]$. Taylor's theorem gives a point ξ between 1 and t such that

$$Q(t) = Q(1) + \frac{1}{2}Q''(\xi)(t-1)^2 \leq Q(1) - \frac{(t-1)^2}{20}.$$

$\square$

Thus the square is a quantitatively isolated maximizer in the certified neighborhood.

Two features of the proof are worth emphasizing. First, the unbounded degenerating regime $t \rightarrow 0$ is eliminated by the exact sublattice inclusion (6.2); no asymptotic numerical calculation is required there. Second, Proposition 4.3 is what turns the verified root of the Lamé matching equation into the geometric accessory parameter of the universal covering.

7. THE RECTANGULAR LANDAU CONSTANT AND A CONCAVITY PROBLEM

Theorem 2.1 has an equivalent scale-invariant form. We use the following result of Sugawa in a self-contained form.

Lemma 7.1 ([Sug02, p. 53, Corollary 4]). *Let $a > 0$, $L = \mathbb{Z} + ai\mathbb{Z}$, and*

$$X = (\mathbb{C} \setminus L)/L.$$

Then the hyperbolic density of the once-punctured rectangular torus X attains its minimum at the unique point

$$\left[\frac{1 + ai}{2} \right].$$

After a similarity, every rectangular lattice is of the form covered by Lemma 7.1; hence the center of a fundamental rectangle is the unique optimal base point for that lattice. For a rectangular lattice Λ , let $d(\Lambda)$ be the radius of the largest Euclidean disc centered at the center of a fundamental rectangle and avoiding Λ , and let $u_\Lambda : \mathbb{D} \rightarrow \mathbb{C} \setminus \Lambda$ be the universal cover sending 0 to that center. Define

$$\mathcal{L}_{\text{rect}} := \inf_{\Lambda \text{ rectangular}} \frac{d(\Lambda)}{|u'_\Lambda(0)|}.$$

The ratio is invariant under similarities.

Corollary 7.2. *The rectangular-lattice restriction of Landau's constant is*

$$\mathcal{L}_{\text{rect}} = \frac{1}{2G} = \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} = \frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(1/4)} = 0.599070117367\dots,$$

and the unique extremal rectangular lattice, up to similarity, is the square lattice.

Proof. In the normalization of Theorem 2.1, $d(\Lambda_a) = 1/2$. Hence

$$\inf_a \frac{d(\Lambda_a)}{f'_a(0)} = \frac{1}{2 \max_a f'_a(0)} = \frac{1}{2G}.$$

Uniqueness follows from Theorem 2.1. The gamma-function form follows from Euler's reflection formula

$$\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}.$$

The identification with the second lemniscate constant is the one recorded in [Fau21, p. 4 and p. 24]. $\square$

The proof above settles the rectangular comparison. It remains natural to ask whether the computer-assisted step can be replaced by a structural inequality. The symmetry of Section 3 reduces such a proof to a particularly simple analytic inequality.

Question 7.3. *Is the function $h(t) = \log r(t)$ concave on $0 < t \leq 1$?*

An affirmative answer would give a short proof of Theorem 2.1. Indeed, differentiating $r(1/t) = r(t)/t$ at $t = 1$ gives $h'(1) = 1/2$. Concavity would imply $h'(t) \geq 1/2$ for $0 < t < 1$, and therefore

$$\frac{d}{dt} \log R(t) = h'(t) - \frac{t}{1+t^2} \geq \frac{1}{2} - \frac{t}{1+t^2} = \frac{(1-t)^2}{2(1+t^2)} > 0.$$

Lemma 7.1 is a spatial extremal statement on each fixed punctured rectangular torus; it does not compare different conformal moduli and therefore does not directly imply the concavity in Question 7.3.

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