

The Lei–Zhang Entropy Theorem and the Krzyż Conjecture

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Entropy

Use normalized arclength on $\mathbb{T}$:

$$dm(e^{i\theta}) = \frac{d\theta}{2\pi}$$

For $f \geq 0$ with $\int_{\mathbb{T}} f \, dm = 1$, define

$$\mathcal{E}(f) = - \int_{\mathbb{T}} f \log f \, dm.$$

By Jensen's inequality,

$$\mathcal{E}(f) \leq 0,$$

with equality only when $f = 1$ almost everywhere.

The uniform density has the largest entropy.

$f(x) = n \mathbb{1}_I$ with $|I| = \frac{1}{n}$ has entropy $-\log(n)$.

Theorem (Jialin Lei–Teng Zhang, 2026)

Let p be a non-constant polynomial, with all zeros on $\mathbb{T}$, and

$$\int_{\mathbb{T}} |p|^2 dm = 1.$$

Then

$$\mathcal{E}(|p|^2) = - \int_{\mathbb{T}} |p|^2 \log |p|^2 dm \leq \log 2 - 1.$$

Equality iff equally spaced zeroes:

$$p(z) = \frac{\omega_1}{\sqrt{2}}(\omega_2 + z^n), \quad |\omega_1| = |\omega_2| = 1.$$

cf. Beckner, 1975

If $\|f\|_{L^2(\mathbb{R})} = 1$,

$$\mathcal{E}(f) + \mathcal{E}(\mathcal{F}(f)) \geq 1 - \log 2.$$

Equality only for Gaussians.

Theorem (Zhang, 2026) – proof of conjecture of A. Baernstein II

Let p have degree $n \geq 1$ and all zeros on $\mathbb{T}$. Put $Q(z) = 1 + z^n$. For $0 \leq s \leq t \leq \infty$,

$$\frac{\|p\|_s}{\|Q\|_s} \leq \frac{\|p\|_t}{\|Q\|_t}.$$

Here, for $0 < s < \infty$,

$$\|p\|_s = \left(\int_{\mathbb{T}} |p|^s dm \right)^{1/s}, \quad \|p\|_0 = \exp \left(\int_{\mathbb{T}} \log |p| dm \right).$$

Differentiating the norm comparison at exponent 2 gives another proof of the entropy inequality.

The Krzyż conjecture (1969)

Let $f : \mathbb{D} \rightarrow \mathbb{D} \setminus \{0\}$ be holomorphic, and write

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k.$$

For every $n \geq 1$,

$$\sup_f |\hat{f}(n)| = \frac{2}{e}.$$

Moreover, extremals only occur for rotations of

$$f(z) = \exp \left(-\frac{1+z^n}{1-z^n} \right).$$

Since f never vanishes, write

$$f = e^{-g}, \quad g : \mathbb{D} \rightarrow \{w : \operatorname{Re} w > 0\}.$$

The Herglotz class $\operatorname{Herg}(\mathbb{D})$

$$\operatorname{Herg}(\mathbb{D}) = \{g : g : \mathbb{D} \rightarrow \text{Right half plane, } g \text{ holomorphic}\}.$$

Original formulation: Maximize the linear functional $\operatorname{Re} \widehat{f}(n)$ over the non-convex set $\mathcal{O}(\mathbb{D}, \mathbb{D} \setminus \{0\})$.

Alternative formulation: maximize the nonlinear functional

$$g \mapsto \operatorname{Re} \widehat{e^{-g}}(n) =: K_n(g)$$

over the convex Herglotz class. $K_n(g)$ depends only on $\widehat{g}(0), \dots, \widehat{g}(n)$.

Herglotz interpolation

Given $a_0, \dots, a_n$, find $g \in \text{Herg}(\mathbb{D})$ so that

$$\widehat{g}(k) = a_k, \quad 0 \leq k \leq n.$$

Say problem is *extremal* if a solution exists, but for every $r > 1$ the data

$$\widehat{g}(k) = r^k a_k, \quad 0 \leq k \leq n,$$

has no solution in $\text{Herg}(\mathbb{D})$.

Extremals for K_n are Herglotz extremals

Hummel, Scheinberg and Zalcman (1977): a Krzyż extremal must give an extremal Herglotz interpolation problem.

Pick's theorem (1916)

Every Herglotz extremal problem has a unique solution of the form

$$g(z) = ia + \sum_{\ell=1}^m w_{\ell} \frac{\tau_{\ell} + z}{\tau_{\ell} - z},$$

where

$$a \in \mathbb{R}, \quad w_{\ell} > 0, \quad \tau_{\ell} \in \mathbb{T} \text{ distinct}, \quad 1 \leq m \leq n.$$

Denote this class by $\mathcal{R}_m$.

The imaginary constant only changes the unimodular factor in f .

The full degree case

The case $m = n$ is the “generic” case.

The entropy theorem will settle the coefficient bound in this case.

The truncated shift

Let

$$H_n^2 = \text{span}\{1, z, \dots, z^n\}, \quad S = P_{H_n^2} M_z|_{H_n^2}.$$

Thus

$$Sz^k = \begin{cases} z^{k+1}, & k < n, \\ 0, & k = n. \end{cases}$$

A finite Toeplitz matrix

Set

$$A_g = \text{Re } g(S) = \text{Re} \left(\sum_{k=0}^n a_k S^k \right).$$

Herglotz interpolation is solvable exactly when $A_g \geq 0$. It is extremal exactly when A_g is also singular.

The kernel in the full degree case

For $g \in \mathcal{R}_m$,

$$\text{rank } A_g = m, \quad \dim \ker A_g = n + 1 - m.$$

If $m = n$, choose the unique $\gamma \in \ker A_g$ with

$$\|\gamma\|_2 = 1, \quad \hat{\gamma}(n) > 0.$$

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$$\begin{aligned} \langle A_g \gamma, \gamma \rangle &= \sum_{\ell=1}^n w_\ell |\gamma(\tau_\ell)|^2 \\ \Rightarrow \gamma(z) &= c \prod_{\ell=1}^n (z - \tau_\ell). \end{aligned}$$

A polynomial with zeros on $\mathbb{T}$

The kernel vector is precisely the kind of polynomial that occurs in the Lei–Zhang theorem.

First variation

At a local maximum of $\operatorname{Re} \widehat{e^{-g}}(n)$,

$$\left. \frac{d}{dt} \right|_{t=0^+} \operatorname{Re} \widehat{e^{-(g+td)}}(n) \leq 0$$

for every admissible direction d .

Admissibility is the matrix condition

$$\operatorname{Re}(g + td)(S) \geq 0$$

for sufficiently small $t \geq 0$.

When $m = n$

There is only one kernel direction, γ .

The first-order supporting condition is just

$$\langle \operatorname{Re} d(S)\gamma, \gamma \rangle \geq 0.$$

Reversal of a polynomial

For $u(z) = \sum_{k=0}^n c_k z^k$, put

$$\tilde{u}(z) = z^n \overline{u(1/\bar{z})} = \sum_{k=0}^n \overline{c_{n-k}} z^k.$$

Reversal here is always relative to the fixed degree n .

Theorem (Agler–McCarthy)

Let $g \in \mathcal{R}_n$, and let γ be its normalized kernel polynomial. The first-order critical-point condition, with $\widehat{e^{-g}}(n) = \eta > 0$, is equivalent to

$$e^{-g(z)} = \eta \gamma(z) \tilde{\gamma}(z) + O(z^{n+1}).$$

Take logarithms

From

$$e^{-g} = \eta \gamma \tilde{\gamma} + O(z^{n+1}),$$

we obtain, through degree n ,

$$-g = \log \eta + \log(\gamma \tilde{\gamma}) + O(z^{n+1}). \quad (*)$$

Since all zeros of γ lie on $\mathbb{T}$,

$$\operatorname{Re} \log(\gamma \tilde{\gamma}) = \log |\gamma|^2 \quad \text{on } \mathbb{T}.$$

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Pair (*) with γ

$$\begin{aligned} 0 = -\langle A_g \gamma, \gamma \rangle &= \log \eta + \int_{\mathbb{T}} |\gamma|^2 \log |\gamma|^2 dm \\ \log \eta &= \mathcal{E}(|\gamma|^2) \\ &\leq \log \frac{2}{e} \quad \text{by Lei - Zhang.} \end{aligned}$$

What remains?

An extremal may have fewer than n boundary atoms. Then $\dim \ker A_g > 1$.

An operator entropy conjecture

Let h be a polynomial of degree $n \geq 1$ with $h(0) \neq 0$, $\|h\|_2 = 1$.

Expand formally at 0: $\log(h\tilde{h}) = \sum_{k=0}^{\infty} b_k z^k$. Define the self-adjoint

Toeplitz matrix on H_n^2 by

$$L_h = \operatorname{Re} \left(\sum_{k=0}^n b_k S^k \right).$$

Conjecture

$$\lambda_{\max}(L_h) \geq 1 - \log 2.$$

A sufficient special case

To prove Krzyż, it would suffice to prove the operator entropy conjecture under the additional hypothesis

$$L_h h = \lambda_{\max}(L_h) h.$$

For a corresponding extremal factorization,

$$e^{-g} = \eta h \tilde{h} + O(z^{n+1}), \quad A_g h = 0,$$

we have

$$A_g = -\log \eta I - L_h \geq 0.$$

The desired coefficient estimate

Thus $-\log \eta = \lambda_{\max}(L_h)$, and the conjecture gives

$$\eta \leq 2/e.$$

Difference: we know $h(\tau_\ell) = 0$, $1 \leq \ell \leq m$, but there are $n - m$ other zeroes which may be off $\mathbb{T}$.

Proof of the Lei–Zhang theorem

Let p have degree n and all zeros on $\mathbb{T}$, with

$$\|p\|_2 = 1.$$

Multiply by a unimodular constant so that $p = \tilde{p}$.

Writing $p(z) = \sum_{j=0}^n a_j z^j$, this says $a_j = \overline{a_{n-j}}$.

The polar factor

Set

$$q(z) = p(z) - \frac{1}{n} z p'(z) = \sum_{j=0}^{n-1} \frac{n-j}{n} a_j z^j.$$

Since $p = \tilde{p}$, reversal relative to degree n gives

$$\tilde{q} = \frac{1}{n} z p', \quad p = q + \tilde{q}.$$

Splitting the entropy integral

For simple zeros of p , show polynomial q has no zeros in $\overline{\mathbb{D}}$. Hence

$$r = \frac{\tilde{q}}{q}$$

is a finite Blaschke product, $r(0) = 0$, and $p = q(1 + r)$.

$$\begin{aligned} \int_{\mathbb{T}} |p|^2 \log |p|^2 dm &= \underbrace{\int_{\mathbb{T}} |p|^2 \log |q|^2 dm}_A \\ &\quad + \underbrace{\int_{\mathbb{T}} |p|^2 \log |1 + r|^2 dm}_B. \end{aligned}$$

Show

$$A \geq -\log 2, \quad B \geq 1.$$

The Jensen term

Since r is inner and $r(0) = 0$,

$$\int_{\mathbb{T}} |1 + r|^2 d\mu = 2.$$

Thus $d\mu = \frac{1}{2}|1 + r|^2 d\mu$ is a probability measure, and

$$\int_{\mathbb{T}} |q|^2 d\mu = \frac{1}{2} \int_{\mathbb{T}} |p|^2 d\mu = \frac{1}{2}.$$

Apply Jensen to the convex function $x \log x$:

$$\begin{aligned} \frac{A}{2} &= \int_{\mathbb{T}} |q|^2 \log |q|^2 d\mu \\ &\geq \left(\int_{\mathbb{T}} |q|^2 d\mu \right) \log \left(\int_{\mathbb{T}} |q|^2 d\mu \right) = -\frac{\log 2}{2}. \end{aligned}$$

The key estimate (Lei–Zhang)

For $n \geq 2$, with p, q, r as above,

$$\int_{\mathbb{T}} |p|^2 \log |1 + r|^2 dm \geq \int_{\mathbb{T}} |p|^2 dm + \frac{2}{n(n-1)} \sum_{j=1}^{n-1} \frac{j(n-j)}{n^2} |a_j|^2.$$

In particular, $B \geq 1$, and equality forces $a_1 = \cdots = a_{n-1} = 0$, so $p(z) = (1 + z^n)/\sqrt{2}$.

Lei–Zhang theorem

For a normalized polynomial with all zeros on $\mathbb{T}$,

$$\mathcal{E}(|p|^2) \leq \log 2 - 1,$$

with equality precisely for equally spaced zeroes.

The remaining question that would prove Krzyż

For normalized polynomials h with $h(0) \neq 0$, is

$$\lambda_{\max}(L_h) \geq 1 - \log 2 ?$$

L_h on H_n^2 defined by

$$L_h = \operatorname{Re} \left[\log(h\tilde{h})(S) \right]$$

Why it is hard

There are local extrema for the Krzyż problem with $m < n$ atoms that are not global extrema. So local methods will not suffice.

Go raibh maith agaibh!
Thank you!