

GAUSSIAN DECAY OF HARMONIC FUNCTIONS AND BACKWARD UNIQUENESS

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ABSTRACT. We construct a planar domain with locally real-analytic boundary for which the supremum of the Gaussian decay exponents of nonzero harmonic functions is $\pi/2$. This gives a negative answer to the proposed equivalence between Gaussian decay of harmonic functions and failure of backward uniqueness for bounded temperatures on arbitrary Dirichlet-regular domains. More precisely, at least one of two explicitly defined dilates violates the equivalence. Regularity is required at finite boundary points; no condition at a compactified point at infinity is imposed.

1. INTRODUCTION

Uniqueness for the heat equation on an unbounded domain depends on both the admissible growth of the solution and the geometry of the domain. The role of growth conditions is already central in the classical work of Tychonoff [Tyc35]. Backward uniqueness without prescribed lateral boundary values leads to a different geometric question: can a bounded solution become identically zero at a finite time? Results for exterior domains were proved by Escauriaza, Seregin, and Šverák [ESS03]. The dependence on the opening of a cone was studied by Li and Šverák [LS12] and Růland [Rul15].

There is a parallel question for harmonic functions: how fast can a nonzero harmonic function decrease at infinity within a given domain? The earlier literature includes Arshon's work on domains of revolution [Ars68]. A proposed connection between these two questions is recorded explicitly in Eremenko's problem note [Ere15, p. 1]. Eremenko attributes the equivalence to the announcement of Gurarii and Matsaev [GM83], while noting the absence of a published proof. Růland [Rul15, Remark 4] also records this proof gap. We address the precise formulation in Eremenko [Ere15, p. 1].

Let $D \subset \mathbb{R}^n$ be a domain. A bounded temperature on $D \times [0, 1]$ is a bounded function

$$u \in C^{2,1}(D \times (0, 1)) \cap C(D \times [0, 1])$$

that satisfies $u_t = \Delta_x u$ in $D \times (0, 1)$. No condition on $\partial D \times [0, 1]$ is included in this definition. We write $I(D)$ and $II(D)$ for the following properties:

- (I) There is a nonzero bounded temperature u on $D \times [0, 1]$ such that $u(x, 1) = 0$ for every $x \in D$.
- (II) There is a nonzero harmonic function v on D such that

$$v(x) = O(e^{-|x|^2}) \quad \text{as } |x| \rightarrow \infty \text{ within } D.$$

The estimate in (II) is uniform: $|v(x)| \leq Ce^{-|x|^2}$ for all $x \in D$ with $|x| \geq R$, for some $C, R > 0$. It does not impose boundedness near the finite boundary. Real-valued functions suffice in both properties, since a nonzero real or imaginary part of a complex-valued function has the same required properties.

The question under consideration can now be stated without an implicit boundary convention.

Problem 1.1. *Are $I(D)$ and $II(D)$ equivalent for every domain $D \subset \mathbb{R}^n$ whose finite boundary points are regular for the Laplace Dirichlet problem?*

We give a negative answer in dimension two. Identify $\mathbb{R}^2$ with $\mathbb{C}$, and let

$$B(w, r) = \{z \in \mathbb{C} : |z - w| < r\}.$$

Define

$$(1.1) \quad \varepsilon_\lambda = \exp[-10(1 + |\lambda|^2)^2], \quad \Lambda = \mathbb{Z} + i\mathbb{Z}, \quad \Omega = \mathbb{C} \setminus \bigcup_{\lambda \in \Lambda} B(\lambda, \varepsilon_\lambda).$$

Each deleted disc lies strictly inside its unit lattice square. The square boundaries form a connected set, and each punctured square is connected to its boundary. Thus Ω is a domain. The discs are locally finite and pairwise disjoint, so every finite boundary point has a neighborhood in which $\partial\Omega$ is a circle. A smaller disc inside the corresponding deleted disc can be chosen tangent at the prescribed boundary point and nowhere

else on $\partial\Omega$. The external ball criterion, as stated by Axler, Bourdon, and Ramey [ABR01, Theorem 11.11, p. 229], gives Dirichlet regularity. The same observations apply to the dilate $2\Omega = \{2z : z \in \Omega\}$.

For a planar domain G , let $\gamma(G)$ denote

$$(1.2) \quad \gamma(G) = \sup \left(\{0\} \cup \left\{ a > 0 : \begin{array}{l} \text{there is } h \in \mathcal{H}(G) \setminus \{0\} \text{ with} \\ h(z) = O(e^{-a|z|^2}) \text{ as } |z| \rightarrow \infty \text{ in } G \end{array} \right\} \right).$$

The following exact value is the main analytic result.

Theorem 1.2. *For the domain Ω in (1.1),*

$$\gamma(\Omega) = \frac{\pi}{2}.$$

The lower bound will be proved by a single nonzero harmonic function h which, for every $0 < a < \pi/2$, satisfies $|h(z)| \leq C_a e^{-a|z|^2}$ on all of Ω . The upper bound excludes every exponent $a > \pi/2$ even when only the asymptotic estimate in (1.2) is assumed. We make no claim about attainment of the endpoint $a = \pi/2$.

The consequence for Problem 1.1 is best stated so that its logical scope is explicit.

Corollary 1.3. *At least one of Ω and 2Ω satisfies exactly one of the properties (I) and (II).*

The reason is a mismatch under dilation. Theorem 1.2 gives (II) for Ω but not for 2Ω , whereas an elementary time restriction and parabolic scaling give $I(\Omega) \Rightarrow I(2\Omega)$. This proves that the equivalence cannot hold on both domains. It does not determine which of them violates which implication, and it does not construct a nonzero temperature on either domain.

The analytic construction uses classical properties of the Weierstrass sigma function for the square lattice; the needed identities are recorded in the NIST Digital Library of Mathematical Functions [Nis26, §23.2]. Real residues of its reciprocal permit a single-valued real harmonic primitive, although no single-valued holomorphic primitive exists. The small holes allow this real primitive to retain Gaussian decay. For the opposite bound, an explicit logarithmic potential and the subharmonic maximum principle rule out Gaussian decay of the holomorphic function obtained from a harmonic gradient. Neither a global solvability theorem for differential operators nor a heat-equation approximation argument is needed.

Organization. Section 2 fixes the sigma-function normalization. Section 3 constructs the decaying harmonic function. Section 4 proves the holomorphic uniqueness statement and completes Theorem 1.2. Section 5 proves Corollary 1.3 and specifies the limitations of the conclusion. In the proofs, C denotes a positive constant that may change between occurrences; dependencies on fixed parameters are indicated when relevant.

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This paper has not yet been carefully polished, and there are currently no plans to submit it for publication.

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2. THE SQUARE-LATTICE SIGMA FUNCTION

Throughout the rest of the paper, set $c = \pi/2$ and $d(z) = \text{dist}(z, \Lambda)$. In particular, $d(z) \leq \sqrt{2}/2$ for every $z \in \mathbb{C}$. Let S be the Weierstrass sigma function for Λ , normalized by $S'(0) = 1$:

$$(2.1) \quad S(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda} \right) \exp \left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2} \right).$$

The product converges normally and defines an entire function with simple zeros precisely at Λ .

For clarity, we specify the classical identities used below. Put $\eta_1 = (S'/S)(1/2)$ and $\eta_3 = (S'/S)(i/2)$. The sigma product, quasi-periodicity formula, and Legendre relation in NIST [Nis26, formulas (23.2.6), (23.2.14), and (23.2.17)] give

$$(2.2) \quad S(z + m + in) = (-1)^{m+n+mn} e^{2(m\eta_1 + n\eta_3)(z + (m+in)/2)} S(z), \quad m, n \in \mathbb{Z},$$

and

$$i\eta_1 - \eta_3 = \pi i.$$

Lemma 2.1. For $\lambda = m + in \in \Lambda$, write $\chi_\lambda = (-1)^{m+n+mn}$. Then

$$(2.3) \quad S(z + \lambda) = \chi_\lambda e^{\pi\lambda z + c|\lambda|^2} S(z).$$

There are constants $C_1, C_2 > 0$ such that

$$(2.4) \quad C_1 d(z) \leq |S(z)| e^{-c|z|^2} \leq C_2 d(z) \quad (z \in \mathbb{C}).$$

At each lattice point,

$$(2.5) \quad S'(\lambda) = \chi_\lambda e^{c|\lambda|^2}, \quad \operatorname{Res}_{z=\lambda} \frac{1}{S(z)} = \chi_\lambda e^{-c|\lambda|^2} \in \mathbb{R}.$$

Proof. The invariance of Λ under multiplication by i , together with (2.1), gives $S(iz) = iS(z)$. Logarithmic differentiation yields $(S'/S)(iz) = -i(S'/S)(z)$ and hence $\eta_3 = -i\eta_1$. The second identity in (2.2) now gives $\eta_1 = \pi/2$ and $\eta_3 = -i\pi/2$. Substitution into the first identity in (2.2) proves (2.3).

Taking absolute values in (2.3) shows that $|S(z)| e^{-c|z|^2}$ is Λ -periodic. On the square $\{z : |\Re z| \leq 1/2, |\Im z| \leq 1/2\}$, the quotient $|S(z)| e^{-c|z|^2} / d(z)$ is positive and continuous away from the lattice point. The simple zero of S gives a positive continuous extension there. Compactness proves (2.4). Differentiating (2.3) at $z = 0$ proves the first identity in (2.5); the second is the residue formula at a simple zero. $\square$

3. A HARMONIC FUNCTION WITH GAUSSIAN DECAY

The real residues in (2.5) are essential: they remove the period obstruction for the real part of a primitive, but not for the primitive itself.

Proposition 3.1. There is a nonzero $h \in \mathcal{H}(\Omega)$ such that, for each $0 < a < c$, a constant $C_a > 0$ satisfies

$$|h(z)| \leq C_a e^{-a|z|^2} \quad (z \in \Omega).$$

Proof. Step 1. A single-valued real primitive. Let $g = 1/S$, fix $z_* = i/2$, and define

$$h_0(z) = \Re \int_{z_*}^z g(\zeta) d\zeta \quad (z \in \mathbb{C} \setminus \Lambda).$$

For any closed, piecewise smooth curve in $\mathbb{C} \setminus \Lambda$, the residue theorem expresses its integral of g as $2\pi i$ times a finite integer linear combination of the real numbers in (2.5). The real part of that integral is zero. Thus h_0 is well-defined and harmonic, with

$$(3.1) \quad (h_0)_x - i(h_0)_y = \frac{1}{S(z)}.$$

Step 2. A common limit at infinity away from the lattice. Fix $\delta = 1/20$. The lower bound in (2.4) implies

$$(3.2) \quad |g(z)| \leq C e^{-c|z|^2} \quad \text{when } d(z) \geq \delta.$$

The line $z = x + i/2$ stays at distance at least $1/2$ from Λ . Integration of (3.2) along its positive half therefore gives a finite limit $\ell = \lim_{x \rightarrow +\infty} h_0(x + i/2)$. Put $h = h_0 - \ell$. For $r \geq 1$, the point $p_r = \sqrt{r^2 - 1/4} + i/2$ has modulus r , and

$$(3.3) \quad |h(p_r)| \leq C \int_{\sqrt{r^2 - 1/4}}^{\infty} e^{-c(s^2 + 1/4)} ds \leq C e^{-cr^2}.$$

Let $|z| = r \geq 2$ and $d(z) \geq \delta$. Join z to p_r along the circle $|\zeta| = r$, replacing each portion inside a disc $B(\lambda, \delta)$ by an arc of its boundary. These discs are disjoint. The number that meet the circle is at most $C(1 + r)$: the disjoint discs $B(\lambda, 1/4)$ centered at the corresponding lattice points lie in an annulus of fixed width and area at most $C(1 + r)$. The modified path therefore has length at most $C(1 + r)$, remains at distance at least δ from Λ , and satisfies $|\zeta| \geq r - 2\delta$ throughout. Equation (3.2) bounds the integral along this path by $C(1 + r)e^{-c(r - 2\delta)^2}$. Together with the endpoint bound (3.3), this proves

$$(3.4) \quad |h(z)| \leq C e^{-c|z|^2 + C_3|z|} \quad \text{when } d(z) \geq \delta,$$

with $C_3 > 0$. Increasing C covers the bounded part of this set.

Step 3. Estimates next to the deleted discs. Suppose $\varepsilon_\lambda < |z - \lambda| < \delta$, and write $z = \lambda + se^{i\theta}$. Join z to $w = \lambda + \delta e^{i\theta}$ along the radial segment. On that segment, $d(\lambda + te^{i\theta}) = t$ and $|\lambda + te^{i\theta}|^2 \geq |\lambda|^2 - 2\delta|\lambda|$. Using (2.4) for the integral and (3.4) for $h(w)$ gives

$$(3.5) \quad |h(z)| \leq C e^{-c|\lambda|^2 + C_4|\lambda|} \left(1 + \log \frac{\delta}{|z - \lambda|} \right).$$

By (1.1), the logarithm in (3.5) is at most $10(1 + |\lambda|^2)^2$. Since $|z - \lambda| < \delta$, replacing $|\lambda|$ by $|z|$ changes the exponent by at most a constant multiple of $1 + |z|$. Equations (3.4) and (3.5) consequently yield

$$(3.6) \quad |h(z)| \leq C(1 + |z|)^4 e^{-c|z|^2 + C_5|z|} \quad (z \in \Omega).$$

For each $a < c$, the function $(1 + r)^4 e^{-(c-a)r^2 + C_5r}$ is bounded on $[0, \infty)$. Thus (3.6) gives the claimed estimate. Finally, (3.1) is unchanged by subtracting ℓ and has a nonzero right-hand side on Ω . Hence h is not constant, in particular not identically zero. $\square$

4. HOLOMORPHIC UNIQUENESS AND THE OPTIMAL EXPONENT

Enlarge each deleted disc by a factor of two and put

$$(4.1) \quad r_\lambda = 2\varepsilon_\lambda, \quad \Omega_* = \mathbb{C} \setminus \bigcup_{\lambda \in \Lambda} B(\lambda, r_\lambda).$$

Only the radii are enlarged; Ω_* is not the dilate 2Ω . The set Ω_* is again a connected domain. Every bounded closed portion of Ω_* is a compact subset of Ω . This elementary separation will let us apply a maximum principle without assuming boundary values on the original circles.

We use the following standard form of that principle. It is stated separately to identify the only potential-theoretic comparison theorem required in the argument.

Lemma 4.1 ([ABR01, Theorem 11.3, p. 225]). *Let $U \subset \mathbb{C}$ be a bounded domain. If $w \in C(\overline{U})$ is subharmonic on U and $w \leq M$ on ∂U , then $w \leq M$ on U .*

Let $\mathcal{O}(G)$ denote the holomorphic functions on an open set G . The next lemma is a Phragmén–Lindelöf argument for the particular deleted discs in (4.1).

Lemma 4.2. *Suppose $F \in \mathcal{O}(\Omega)$ and there are $b, C, R > 0$ such that*

$$(4.2) \quad |F(z)| \leq C e^{-b|z|^2} \quad (z \in \Omega_*, |z| \geq R).$$

Then $F \equiv 0$ on Ω .

Proof. Step 1. A convergent logarithmic potential. The radii in (4.1) satisfy

$$\log(1/r_\lambda) = 10(1 + |\lambda|^2)^2 - \log 2 > 0.$$

For $|\lambda| \geq L$, where $L \geq 2$ will be fixed shortly, set

$$a_\lambda = \frac{4 \log(2 + |\lambda|)}{\log(1/r_\lambda)}.$$

Counting lattice points in annuli of unit width gives

$$(4.3) \quad \sum_{|\lambda| \geq L} a_\lambda (1 + \log(1 + |\lambda|)) < \infty.$$

Indeed, the contribution of $k \leq |\lambda| < k+1$ is at most $C(1+k)^{-3} \log^2(2+k)$, a summable sequence. Choose L so large that $A = \sum_{|\lambda| \geq L} a_\lambda < 1$, and define

$$(4.4) \quad q(z) = \log |z| - K + \sum_{|\lambda| \geq L} a_\lambda \log \frac{|z - \lambda|}{1 + |\lambda|}, \quad z \in \Omega_*.$$

The constant $K \geq 0$ will be chosen in Step 2. The origin lies in a deleted disc, so the first logarithm is harmonic on Ω_* . For $|z| \leq M_0$ and $|\lambda| > 2M_0 + 1$, the absolute value of the logarithmic summand before multiplication by a_λ is at most $CM_0/|\lambda|$. Thus (4.3) proves absolute local uniform convergence of (4.4). Its sum is harmonic and extends continuously across every finite boundary circle of Ω_* .

Step 2. Boundary signs and growth. The inequality $|z - \lambda|/(1 + |\lambda|) \leq 1 + |z|$, the positivity of a_λ , and $K \geq 0$ imply

$$(4.5) \quad q(z) \leq (1 + A) \log(1 + |z|) \quad (z \in \Omega_*).$$

If $|\lambda| \geq L$ and $|z - \lambda| = r_\lambda$, isolate the λ -term of (4.4). Since $r_\lambda < 1$, the remaining terms are bounded above by their coefficients times $\log(2 + |\lambda|)$. Therefore

$$q(z) \leq (1 + A) \log(2 + |\lambda|) - a_\lambda \log(1/r_\lambda) - K = (A - 3) \log(2 + |\lambda|) - K < 0.$$

There are only finitely many circles with $|\lambda| < L$. The local uniform convergence of (4.4) makes its restriction to each such circle continuous. Taking K sufficiently large now ensures

$$(4.6) \quad q \leq 0 \quad \text{on every boundary circle of } \Omega_*.$$

On the other hand, when $d(z) \geq 1/4$, each logarithmic term in (4.4) is at least $-\log 4 - \log(1 + |\lambda|)$. By (4.3),

$$(4.7) \quad q(z) \geq \log |z| - C_0 \quad \text{when } d(z) \geq 1/4.$$

In particular, (4.7) along $z = x + i/2$ shows that $\{z \in \Omega_* : q(z) > 0\}$ is a nonempty open set.

Step 3. The maximum principle. Because F is holomorphic on Ω , it is continuous on every bounded closed portion of Ω_* . Together with (4.2), this shows that

$$M = \max\{1, C, \sup_{\Omega_*} |F|\}$$

is finite. Fix $s > 0$ and set $W_s = |F|^2 e^{2sq}$ on Ω_* . This is a continuous subharmonic function: locally, q has a harmonic conjugate $\tilde{q}$, and $W_s = |F e^{s(q+i\tilde{q})}|^2$ is the squared modulus of a holomorphic function. By (4.6), $W_s \leq M^2$ on every boundary circle. Also, (4.2) and (4.5) give

$$W_s(z) \leq C^2 e^{-2b|z|^2} (1 + |z|)^{2s(1+A)} \longrightarrow 0 \quad \text{as } |z| \rightarrow \infty \text{ in } \Omega_*.$$

For a positive integer N , let

$$Q_N = \{z : |\Re z| < N + 1/2, |\Im z| < N + 1/2\}.$$

The boundary of Q_N is at distance at least $1/2$ from Λ and therefore meets no deleted disc. The domain $Q_N \cap \Omega_*$ has only finitely many circular holes; W_s is continuous on its closure. For all sufficiently large N , $W_s \leq M^2$ on its outer square boundary as well. Lemma 4.1 gives $W_s \leq M^2$ throughout $Q_N \cap \Omega_*$. Letting N increase yields

$$(4.8) \quad |F(z)| \leq M e^{-sq(z)} \quad (z \in \Omega_*).$$

The number M is independent of s . At every point where $q(z) > 0$, letting $s \rightarrow \infty$ in (4.8) gives $F(z) = 0$. The identity theorem on the connected domain Ω completes the proof. $\square$

We will apply Lemma 4.2 to a holomorphic multiple of a harmonic gradient. The required local bound is the first-order case of the standard Cauchy estimates for harmonic functions.

Lemma 4.3 ([ABR01, Theorem 2.4, p. 33]). *There is an absolute constant $C_* > 0$ such that every bounded real-valued harmonic function v on $B(z, r) \subset \mathbb{C}$ satisfies*

$$|v_x(z) - i v_y(z)| \leq \frac{C_*}{r} \sup_{B(z, r)} |v|.$$

The enlargement in (4.1) supplies the discs needed for Lemma 4.3, even when z lies very close to a small hole.

Proof of Theorem 1.2. Proposition 3.1 and the definition (1.2) give $\gamma(\Omega) \geq c$. To prove the reverse inequality, suppose $v \in \mathcal{H}(\Omega)$ satisfies $|v(z)| \leq C e^{-a|z|^2}$ for $|z| \geq R$, where $a > c$. Choose a_1 with $c < a_1 < a$.

Step 1. A local gradient estimate on Ω_* . For $z \in \Omega_*$, set

$$(4.9) \quad \rho(z) = \min\{1/8, d(z)/4\}.$$

For every $\lambda \in \Lambda$, the definition of Ω_* gives $|z - \lambda| > 2\varepsilon_\lambda$. Hence

$$|z - \lambda| - \varepsilon_\lambda \geq \frac{1}{2}|z - \lambda| \geq \frac{1}{2}d(z) > \rho(z).$$

Thus $\overline{B(z, \rho(z))} \subset \Omega$. For sufficiently large $|z|$, the whole disc lies in $\{|w| \geq R\}$, and $|w| \geq |z| - 1/8$ there. Lemma 4.3 then gives

$$(4.10) \quad |v_x(z) - i v_y(z)| \leq \frac{C}{\rho(z)} e^{-a(|z| - 1/8)^2} \leq \frac{C'}{\rho(z)} e^{-a_1|z|^2}.$$

The last inequality follows because $-(a - a_1)r^2 + (a/4)r - a/64$ is bounded above for $r \geq 0$.

Step 2. Reduction to holomorphic uniqueness. The function

$$(4.11) \quad F(z) = S(z)(v_x(z) - i v_y(z))$$

is holomorphic on Ω , since $\Delta v = 0$ implies the Cauchy–Riemann equations for $v_x - iv_y$. Equation (4.9) and $d(z) \leq \sqrt{2}/2$ give $d(z)/\rho(z) \leq 6$. The upper bound in (2.4) and (4.10) therefore imply

$$(4.12) \quad |F(z)| \leq Ce^{-(a_1-c)|z|^2} \quad (z \in \Omega_*, |z| \text{ sufficiently large}).$$

Since $a_1 - c > 0$, (4.12) satisfies the hypothesis of Lemma 4.2. Thus $F \equiv 0$. The sigma function has no zeros on Ω , so (4.11) gives $v_x = v_y = 0$ there. The domain is connected, and the decay of v forces this constant function to be zero. No exponent $a > c$ is admissible in (1.2). Hence $\gamma(\Omega) = c = \pi/2$. $\square$

5. DILATION AND BACKWARD UNIQUENESS

The parabolic part of the argument uses only the restriction of an existing solution to a suitable time interval. In particular, it requires neither extension across a terminal time nor any regularity of the lateral boundary.

Lemma 5.1. *For every domain $D \subset \mathbb{R}^n$ and every $s \geq 1$,*

$$(5.1) \quad \mathbf{I}(D) \implies \mathbf{I}(sD).$$

Proof. Let u witness $\mathbf{I}(D)$, and put $b = \sup\{t \in [0, 1] : u(\cdot, t) \not\equiv 0\}$. Continuity and nontriviality imply $b > 0$. If $b < 1$, the solution is zero for $t > b$, so continuity gives $u(\cdot, b) = 0$; if $b = 1$, the same equality is the terminal condition.

Set $\tau = s^{-2}$ and $t_0 = \max\{0, b - \tau\}$. The interval $[t_0, t_0 + \tau]$ is contained in $[0, 1]$, its right endpoint is at least b , and it contains a nonzero time slice because $t_0 < b$ and b is the supremum of the nonzero times. Define

$$U(X, t) = u(X/s, t_0 + t/s^2), \quad X \in sD, \quad 0 \leq t \leq 1.$$

Then $U_t = s^{-2}u_t = s^{-2}\Delta_x u = \Delta_X U$ on the open time interval. The required endpoint continuity and boundedness are inherited from u . The right endpoint is a zero time, whereas U is not identically zero. Thus U witnesses (5.1). $\square$

For harmonic functions the dilation law follows directly from their scaling and from the uniform meaning of the asymptotic estimate in (1.2):

$$(5.2) \quad \gamma(sG) = s^{-2}\gamma(G) \quad (s > 0).$$

Indeed, v on sG corresponds to $h(z) = v(sz)$ on G , and an exponent a for v becomes the exponent as^2 for h ; the correspondence is reversible.

Proof of Corollary 1.3. Proposition 3.1, with $a = 1 < c$, proves $\mathbf{II}(\Omega)$. Theorem 1.2 and (5.2) give $\gamma(2\Omega) = \pi/8 < 1$, so $\mathbf{II}(2\Omega)$ is false.

If $\mathbf{I}(\Omega)$ is false, then Ω satisfies (II) but not (I). If $\mathbf{I}(\Omega)$ is true, Lemma 5.1, with $s = 2$, gives $\mathbf{I}(2\Omega)$, while (II) is false on 2Ω . These cases prove the corollary. $\square$

The conclusion concerns the uniform quantifier over all Dirichlet-regular domains in Problem 1.1. It does not assert that both implications fail, nor does it settle the backward uniqueness question on a prescribed cone. The boundary of Ω is locally real analytic, but the hole radii tend to zero; no uniform exterior ball radius or regularity at a compactified point at infinity is claimed.

Requiring the estimate in (II) on the whole domain does not repair the equivalence: Proposition 3.1 already provides that stronger estimate on Ω , and its failure on 2Ω follows even from the weaker asymptotic requirement. More generally, replacing the fixed exponent 1 by any prescribed $a > 0$ still gives no universal criterion. To see this, take $G = \sqrt{\pi/(4a)}\Omega$ in (5.2). Then $\gamma(G) = 2a$ and $\gamma(2G) = a/2$, and the same two-case argument applies. Statements with a different quantifier on the decay exponent, or with additional geometric hypotheses, are separate questions.

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