BORCEA'S 2-VARIANCE CONJECTURE

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Abstract. In this paper, we prove Borcea's 2-variance conjecture.

1. Introduction

Very recently, the geometry of polynomials has witnessed a series of major advances.

- Mazur [Maz26] obtained a computer-verified proof of Sendov's conjecture. Tao [Tao26] subsequently digested and reorganized the argument, substantially streamlining the proof and bringing its underlying mathematical structure to the foreground. In this process, Tao also found the same ideas can prove the stronger Phelps–Rodriguez conjecture [PR72].
- Building on the ideas developed in Tao's treatment of Sendov's conjecture [Tao26], the author [Zha26b] established a quantitative strengthening of Sendov's conjecture named the *quadratic Tang–Zhang inequality.*
- A counterexample of Smale's mean value conjecture was found in [Ada26].

A closer look at the proofs of Sendov's conjecture and the quadratic Tang–Zhang inequality shows that they follow essentially the same pattern:

assuming that a counterexample exists: direct arguments in low degrees, analytic estimates in high degrees, and computer-assisted verification for the remaining intermediate cases.

This observation naturally suggests that the same general strategy may be effective for several related problems, including Schmeisser's conjecture [Sch77], Borcea's variance conjecture [KPPSS11] and related generalizations [TZ25, Tao26, Zha26a]. In this paper, we focus on Borcea's variance conjecture.

Let P be a complex polynomial of degree $n \geq 2$ with zeros $z_1, \ldots, z_n$, counted with multiplicity. For $0 < p \leq \infty$, define the p-variance of the zero set of P by

$$\sigma_p(P) = \inf_{c \in \mathbb{C}} \left(\frac{1}{n} \sum_{j=1}^{n} |z_j - c|^p \right)^{\frac{1}{p}} \qquad (0 < p < \infty),$$
$$\sigma_\infty(P) = \inf_{c \in \mathbb{C}} \max_{1 \leq j \leq n} |z_j - c|.$$

By the monotonicity of normalized ℓ^p-norms, $\sigma_p(P)$ is nondecreasing in p; that is,

$$\sigma_p(P) \leq \sigma_q(P) \qquad (0 < p < q \leq \infty).$$

Certain values of $\sigma_p(P)$ are of particular importance, see [KPPSS11, §2]. The quantity $\sigma_1(P)$ is called the *mean deviation* of the zero set of P. For $p = 2$, by the basic inner space techniques, the minimizing center is the centroid $c_P = n^{-1} \sum_{j=1}^{n} z_j$, and hence

$$\sigma_2(P)^2 = \frac{1}{n} \sum_{j=1}^{n} |z_j - c_P|^2.$$

Thus $\sigma_2(P)^2$ is the variance of the zeros of P, while $\sigma_2(P)$ is their standard deviation. For $p = \infty$, $\sigma_\infty(P)$ is called the *Chebyshev radius* of the zero set of P; equivalently, it is the

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radius of the smallest closed disk containing all the zeros of P. Any minimizing center in the definition of $\sigma_\infty(P)$ is called a *Chebyshev center* of the zero set.

Motivated by his work on Sendov's conjecture [Bor96a, Bor96b, Bor98, Bor06, Bor07], Borcea proposed a family of p-variance conjectures in which the spread of the zeros is measured by centered moments rather than by the radius of a containing disk; see [KPPSS11, Conjecture 1].

Conjecture 1.1 (Borcea p-variance conjectures)**.** *Let P be a complex polynomial of degree $n \geq 2$ and $1 \leq p \leq \infty$. For every zero a of P, there exists a critical point ζ of P such that*

$$|a - \zeta| \leq \sigma_p(P).$$

Since $\sigma_p(P)$ is nondecreasing in p, the case $p = 1$ is the strongest form of Conjecture 1.1. At the other extreme, the limiting case $p = \infty$, which is the weakest member of the family, is equivalent to Sendov's conjecture.

Conjecture 1.1 is known in several special cases:

- $n \leq 3$ and $p = 1$ [KPPSS11, §4];
- P has at most three distinct zeros and $p = 2$ [KPPSS11, Theorem 3.2];
- all zeros of P are real and $p = 2$; Borcea (see [KPPSS11, Theorem 6.1]) obtained the stronger result that the radius $\sigma_2(P)$ can be reduced by the factor $1/\sqrt{n-1}$.

Our main result settles the case $n \geq 2$ and $p = 2$ in Conjecture 1.1.

Theorem 1.2. *Let P be a complex polynomial of degree $n \geq 2$. For every zero a of P, there exists a critical point ζ of P such that*

$$|a - \zeta| \leq \sigma_2(P). \tag{1.1}$$

The constant 1 in (1.1) is sharp.

As an immediate consequence of Theorem 1.2 (see also [KPPSS11, §6.3]), we obtain the following corollary. Indeed, Malamud [Mal05] related the zeros and critical points of a polynomial to the spectra of a normal matrix and one of its principal submatrices. In the circulant setting, Kushel and Tyaglov [KT16, Theorem 2.1] showed that if C is an $n \times n$ circulant matrix with characteristic polynomial P, then the spectrum of its leading $(n-1) \times (n-1)$ principal submatrix consists precisely of the critical points of P, counted with multiplicity.

Corollary 1.3. *Assume that $n \geq 3$ and $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Then the $(n-1) \times (n-1)$ Toeplitz matrix*

$$B = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$

has at least one eigenvalue λ satisfying

$$\left| \lambda - \sum_{k=0}^{n-1} a_k \right|^2 \leq \sum_{k=1}^{n-1} |a_k|^2.$$

Sketch of the proof. We briefly describe the proof of Theorem 1.2. After normalization, a hypothetical counterexample is described by the reciprocal critical-point coordinates

$$q_j = \frac{1}{a - \zeta_j}, \qquad \mu = \frac{1}{n-1} \sum_{j=1}^{n-1} q_j, \qquad s = \frac{1}{n-1} \sum_{j=1}^{n-1} |q_j|^2,$$

with $|q_j| < 1$. Schoenberg's inequality and several Cauchy–Schwarz estimates reduce the geometry of the critical points to quantitative restrictions on a, $|\mu|$, and s.

We then study $F(t) = \prod_{j=1}^{n-1}(1 - atq_j)$. Two polynomial identities, together with direct product estimates and a centered interpolation between $q_j = \mu$ and the actual configuration, give explicit scalar inequalities that every normalized counterexample must satisfy.

These inequalities exclude all $n \geq 100001$ by elementary analytic estimates. For $4 \leq n \leq 100000$, we first obtain additional product and coefficient bounds and then reduce the problem to a compact box in the parameters $(a, |\mu|, s)$. A finite rational subdivision of this box is excluded by exact-arithmetic certificates with directed rounding. Together with the known cases $n = 2, 3$, this proves the theorem.

Organization of the paper. In Section 2, we normalize a hypothetical counterexample and introduce the reciprocal coordinates and second-moment notation. Section 3 establishes the basic reciprocal-moment inequalities, polynomial identities, and product bounds. In Section 4, we develop the direct and centered integral estimates used throughout the proof. Section 5 excludes all degrees $n \geq 100001$ by purely analytic estimates. Sections 6 and 7 derive the additional product and coefficient bounds needed for the remaining degrees, including a separate analytic treatment of configurations near $|\mu| = 1$. Section 8 reduces the remaining problem to a compact three-parameter box and completes the proof by an exact finite verification. Finally, Appendix A gives the mathematical specification of the interval tests used by the certificates, while Appendix B describes the exact arithmetic, certificate format, and reproduction procedure.

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2. Reduction and notation

We first normalize the problem and introduce the notation that will be used throughout the proof. Suppose that (1.1) fails at a zero of a polynomial of degree $n \geq 4$. The variance is positive and the prescribed zero is simple. Translate the centroid to zero, divide the zeros by $\sigma_2(P)$, rotate the prescribed zero to the nonnegative real axis, and make the polynomial monic. The resulting polynomial satisfies

$$\begin{aligned} P(z) &= (z - a)\prod_{j=1}^{n-1}(z - z_j), \qquad a \geq 0, \\ a + \sum_{j=1}^{n-1} z_j &= 0, \qquad a^2 + \sum_{j=1}^{n-1} |z_j|^2 = n. \end{aligned} \tag{2.1}$$

Let $\zeta_1, \ldots, \zeta_{n-1}$ be its critical points.

The classical Schoenberg inequality states that if $z_1, \ldots, z_n$ are the zeros of a complex polynomial P, and $\zeta_1, \ldots, \zeta_{n-1}$ are its critical points, all counted with multiplicity, then,

provided that the centroid of the zeros is the origin, i.e. $\sum_{j=1}^{n} z_j = 0$, one has

$$\sum_{k=1}^{n-1} |\zeta_k|^2 \le \frac{n-2}{n} \sum_{j=1}^{n} |z_j|^2, \tag{2.2}$$

where equality holds if and only if all zeros are collinear in the complex plane. This was conjectured by Schoenberg [Sch86] and confirmed independently by Pereira [Per03] and Malamud [Mal05]. It was subsequently reproved by Cheung–Ng [CN06] and Kushel–Tyaglov [KT16] using distinct approaches. (2.2) has since inspired a series of higher-order generalizations, collectively referred to as *Schoenberg type inequalities*, see [BS99, CN06, KT16, LXZ21, GGTW25, Tan25, TZ25, Zha26c, Zha26d]. As an interesting application of (2.2), the author [Zha26a, Zha27] presented some improvements of Schmeisser's conjecture [Sch77] for the barycenter case.

Coefficient comparison and the classical Schoenberg inequality (2.2) give

$$\frac{1}{n-1} \sum_{j=1}^{n-1} \zeta_j = 0, \qquad V_* := \frac{1}{n-1} \sum_{j=1}^{n-1} |\zeta_j|^2 \le V_n := 1 - \frac{1}{n-1}. \tag{2.3}$$

A counterexample has $|a - \zeta_j| > 1$ for every j, so (2.3) excludes $a = 0$. The Cauchy–Schwarz inequality in (2.1) gives $a^2 \le n - 1$.

In the rest of the paper, a *normalized counterexample* means a polynomial satisfying (2.1) with $a > 0$ and $|a - \zeta_j| > 1$ for every j. Unless stated otherwise, the following notation refers to such a counterexample.

Notation. We shall use the following notation throughout the argument. For the quantities already appearing in Tao's exposition [Tao26], we keep the same notation, namely a, z_j, ζ_j, q_j, x, y, $F(t)$, and J. We also retain from [Zha26b] the second-moment notation s, w_j, v, and $\beta_s(t)$. The remaining quantities μ, V, ρ, e_k, and u_+ are introduced for the purposes of the present argument.

- $q_j := \dfrac{1}{a - \zeta_j}$, where $0 < |q_j| < 1$ for $1 \le j \le n-1$.
- $\mu := \dfrac{1}{n-1} \sum_{j=1}^{n-1} q_j = x + iy$, where $x, y \in \mathbb{R}$.
- $w_j := q_j - \mu$ for $1 \le j \le n-1$. Then $\sum_{j=1}^{n-1} w_j = 0$.
- $s := \dfrac{1}{n-1} \sum_{j=1}^{n-1} |q_j|^2$, where $0 < s < 1$.
- $v := \dfrac{1}{n-1} \sum_{j=1}^{n-1} |w_j|^2 = s - |\mu|^2$.
- V denotes a specified upper bound for V_* with $0 \le V \le 1$. Its default value is V_n from (2.3); individual arguments may use $V = 1$ or a sharper bound.
- $\rho := 2ax - a^2 s$.
- $\beta_s(t) = 1 - 2axt + a^2 s t^2$, where $0 \le t \le 1$.
- $F(t) := \prod_{j=1}^{n-1} (1 - atq_j)$.
- $J := \prod_{j=1}^{n-1} z_j q_j$.
- $e_k(\boldsymbol{\xi}) := \sum_{1 \le j_1 < \cdots < j_k \le n-1} \xi_{j_1} \cdots \xi_{j_k}$ for $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{C}^{n-1}$, and $e_0(\boldsymbol{\xi}) := 1$. Empty sums are zero and empty products are one.
- $u_+ := \max\{u, 0\}$ for $u \in \mathbb{R}$.

3. Reciprocal moments and polynomial identities

Lemma 3.1. *Every normalized counterexample satisfies*

$$\rho \geq |\mu|^2 + (1-V)(1-s), \qquad |\mu - a|^2 \leq (a^2 + V - 1)(1-s), \tag{3.1}$$

$$|1 - a\mu|^2 \leq Vv, \qquad s(a^2 + V) \geq 1, \tag{3.2}$$

$$\beta_s(1) \leq (a^2 + V)v. \tag{3.3}$$

Proof. Since $q_j = (a - \overline{\zeta_j})|q_j|^2$ and $\sum_{j=1}^{n-1} \zeta_j = 0$,

$$\mu - as = \frac{1}{n-1} \sum_{j=1}^{n-1} \overline{\zeta_j}(1 - |q_j|^2).$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |\mu - as|^2 &\leq \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |\zeta_j|^2 (1 - |q_j|^2) \right) \left(\frac{1}{n-1} \sum_{j=1}^{n-1} (1 - |q_j|^2) \right) \\ &= \big(V_* - \beta_s(1)\big)(1-s) \\ &\leq \big(V - \beta_s(1)\big)(1-s), \end{aligned}$$

which is equivalent to the first inequality in (3.1).

Since $|\mu - a|^2 = |\mu|^2 - \rho + a^2(1-s)$, the first inequality in (3.1) yields

$$|\mu - a|^2 \leq (a^2 + V - 1)(1-s),$$

which proves the second inequality in (3.1).

Since $(a - \zeta_j)q_j = 1$ and $\sum_{j=1}^{n-1} \zeta_j = 0$, we have

$$a\mu - 1 = \frac{1}{n-1} \sum_{j=1}^{n-1} \zeta_j (q_j - \mu).$$

Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned} |1 - a\mu|^2 &\leq \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |\zeta_j|^2 \right) \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |q_j - \mu|^2 \right) \\ &= V_* v \leq Vv, \end{aligned}$$

which proves the first inequality in (3.2).

Since $q_j(a - \zeta_j) = 1$, the Cauchy–Schwarz inequality gives

$$\begin{aligned} 1 &= \left| \frac{1}{n-1} \sum_{j=1}^{n-1} q_j (a - \zeta_j) \right|^2 \\ &\leq \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |q_j|^2 \right) \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |a - \zeta_j|^2 \right) \\ &= s(a^2 + V_*) \qquad \text{since } \sum_{j=1}^{n-1} \zeta_j = 0 \\ &\leq s(a^2 + V), \end{aligned}$$

which proves the second inequality in (3.2).

Finally, since $v = s - |\mu|^2$ and $\mu = x + iy$,

$$\begin{aligned} \beta_s(1) &= 1 - 2ax + a^2 s \\ &= |1 - a\mu|^2 + a^2 v \\ &\leq (a^2 + V)v \qquad \text{by the first inequality in (3.2)}, \end{aligned}$$

which proves (3.3). □

Lemma 3.2. *For every normalized counterexample, define*

$$\mathcal{R} := a^2 \int_0^1 t \sum_{j=1}^{n-1} q_j^2 \prod_{\substack{1\le k\le n-1\\ k\ne j}} (1 - atq_k)\,\mathrm{d}t.$$

Then

$$(-1)^{n-1} J = n \int_0^1 F(t)\,\mathrm{d}t. \tag{3.4}$$

$$F(1) + (n-1)a\mu \int_0^1 F(t)\,\mathrm{d}t + \mathcal{R} = 1. \tag{3.5}$$

Moreover,

$$(-1)^{n-1} a\mu J + (1-a\mu)^n - na\mu \int_0^1 \left(F(t) - (1-a\mu t)^{n-1}\right)\mathrm{d}t = 1. \tag{3.6}$$

Proof. For the identity (3.4), see [Tao26, Lemma 6(iii), equation (9)] or [Zha26b, Lemma 3.1(i)].

For the differential identity (3.5), see [Zha26b, Lemma 3.1(iii)], with $x+iy = \mu$ and $R = \mathcal{R}$.

It remains only to prove (3.6). Since

$$na\mu \int_0^1 (1-a\mu t)^{n-1}\,\mathrm{d}t = 1 - (1-a\mu)^n,$$

where the identity remains valid when $a\mu = 0$, (3.4) gives

$$\begin{aligned}(-1)^{n-1} a\mu J &= na\mu \int_0^1 F(t)\,\mathrm{d}t \\ &= na\mu \int_0^1 \left(F(t) - (1-a\mu t)^{n-1}\right)\mathrm{d}t + 1 - (1-a\mu)^n.\end{aligned}$$

Rearranging yields (3.6). □

We shall use both a root product bound and a critical-point product bound.

Lemma 3.3. *Define*

$$\Phi_{n-1}(a) = a\left(1 + \frac{1-a^2}{n-1}\right)^{(n-1)/2}, \qquad \psi(a) = a\exp\left(\frac{1-a^2}{2}\right).$$

Every normalized counterexample satisfies

$$|a\mu J| \le \Phi_{n-1}(a)|\mu| \prod_{j=1}^{n-1} |q_j|, \qquad \text{where } \Phi_{n-1}(a) \le \min\{1, \psi(a)\}. \tag{3.7}$$

$$\left[\frac{n}{n-1}(1+a^2)s\right]^{n-1} \ge n^2. \tag{3.8}$$

For fixed a, $\Phi_{n-1}(a)$ is nondecreasing in n on its domain $n-1 \ge a^2$. For fixed n, it increases up to $a = 1$ and decreases thereafter.

Proof. We first prove (3.7). By the normalization (2.1), $\sum_{j=1}^{n-1} |z_j|^2 = n - a^2$. Hence the arithmetic–geometric mean inequality gives

$$\prod_{j=1}^{n-1} |z_j| \le \left(\frac{1}{n-1}\sum_{j=1}^{n-1} |z_j|^2\right)^{(n-1)/2} = \left(\frac{n-a^2}{n-1}\right)^{(n-1)/2}.$$

Multiplying by a, we obtain

$$a\prod_{j=1}^{n-1} |z_j| \le a\left(\frac{n-a^2}{n-1}\right)^{(n-1)/2} = a\left(1 + \frac{1-a^2}{n-1}\right)^{(n-1)/2} = \Phi_{n-1}(a). \tag{3.9}$$

We next bound $\Phi_{n-1}(a)$. Applying the arithmetic–geometric mean inequality to the n nonnegative numbers

$$a^2, \quad \frac{n-a^2}{n-1}, \quad \dots, \quad \frac{n-a^2}{n-1}$$

gives

$$a^2\left(\frac{n-a^2}{n-1}\right)^{n-1} \le 1.$$

The left-hand side is precisely $\Phi_{n-1}(a)^2$, and hence $\Phi_{n-1}(a) \le 1$.

On the other hand, since $1+\frac{1-a^2}{n-1} > 0$, the elementary inequality $\log(1+u) \le u$ when $u > -1$ gives

$$\begin{aligned}\log \Phi_{n-1}(a) &= \log a + \frac{n-1}{2}\log\left(1+\frac{1-a^2}{n-1}\right)\\ &\le \log a + \frac{1-a^2}{2} = \log \psi(a).\end{aligned}$$

Thus $\Phi_{n-1}(a) \le \psi(a)$, and consequently

$$\Phi_{n-1}(a) \le \min\{1, \psi(a)\}.$$

This gives the second inequality in (3.7).

Finally, by the definition of J,

$$|a\mu J| = a|\mu| \prod_{j=1}^{n-1} |z_j| \prod_{j=1}^{n-1} |q_j|.$$

Combining this identity with (3.9) yields

$$|a\mu J| \le \Phi_{n-1}(a)|\mu| \prod_{j=1}^{n-1} |q_j|,$$

which proves the first inequality in (3.7).

We now prove (3.8). Using $\sum_{j=1}^{n-1} z_j = -a$ and $\sum_{j=1}^{n-1} |z_j|^2 = n-a^2$, we obtain

$$\sum_{j=1}^{n-1} |a-z_j|^2 = n(1+a^2).$$

Thus

$$\frac{1}{n-1}\sum_{j=1}^{n-1} |a-z_j|^2 = \frac{n(1+a^2)}{n-1}.$$

The arithmetic–geometric mean inequality gives

$$\prod_{j=1}^{n-1} |a-z_j|^2 \le \left(\frac{n(1+a^2)}{n-1}\right)^{n-1}. \tag{3.10}$$

Recall $P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$. Then

$$P'(a) = \prod_{j=1}^{n-1}(a-z_j).$$

On the other hand, $P'(z) = n\prod_{j=1}^{n-1}(z-\zeta_j)$, and therefore, using $q_j = (a-\zeta_j)^{-1}$,

$$|P'(a)|^2 = n^2 \prod_{j=1}^{n-1} |a-\zeta_j|^2 = \frac{n^2}{\prod_{j=1}^{n-1}|q_j|^2}.$$

It follows that

$$\prod_{j=1}^{n-1} |a - z_j|^2 = \frac{n^2}{\prod_{j=1}^{n-1} |q_j|^2}. \tag{3.11}$$

Moreover, by the arithmetic–geometric mean inequality and the definition of s,

$$\prod_{j=1}^{n-1} |q_j|^2 \le \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |q_j|^2 \right)^{n-1} = s^{n-1}. \tag{3.12}$$

Combining (3.10), (3.11), and (3.12) gives

$$\frac{n^2}{s^{n-1}} \le \frac{n^2}{\prod_{j=1}^{n-1} |q_j|^2} = \prod_{j=1}^{n-1} |a - z_j|^2 \le \left(\frac{n(1+a^2)}{n-1} \right)^{n-1}.$$

Therefore

$$\left[\frac{n}{n-1} (1+a^2) s \right]^{n-1} \ge n^2,$$

which proves (3.8).

It remains to verify the two monotonicity assertions. Fix $a > 0$ and write $m = n - 1$, and $b = 1 - a^2$. Then

$$\log \Phi_m(a) = \log a + \frac{m}{2} \log \left(1 + \frac{b}{m} \right).$$

Regarding m as a continuous variable, differentiation gives

$$\frac{\partial}{\partial m} \log \Phi_m(a) = \frac{1}{2} \left[\log \left(1 + \frac{b}{m} \right) - \frac{b/m}{1 + b/m} \right].$$

Notice that for every $u > -1$,

$$h(u) = \log(1+u) - \frac{u}{1+u} \ge 0.$$

Thus

$$\frac{\partial}{\partial m} \log \Phi_m(a) \ge 0,$$

and hence, for fixed a, $\Phi_{n-1}(a)$ is nondecreasing in n throughout its domain.

Finally, for fixed n,

$$\frac{\mathrm{d}}{\mathrm{d}a} \log \Phi_{n-1}(a) = \frac{n(1-a^2)}{a(n-a^2)}.$$

Since $a > 0$ and $a^2 \le n - 1 < n$, the denominator is positive. Thus the derivative has the sign of $1 - a^2$. Therefore $\Phi_{n-1}(a)$ is increasing for $0 < a \le 1$ and decreasing for $1 \le a \le \sqrt{n-1}$, as claimed. □

4. Integral estimates

The estimates in this section will be used twice: first with simple exponential majorants, and then on rational intervals. Write

$$c_n = \left(\frac{n-1}{n-2} \right)^{(n-2)/2}, \quad \widetilde{c}_n = c_n^2, \qquad d_n = \left(\frac{n-1}{n-3} \right)^{(n-3)/2}, \quad \widetilde{d}_n = d_n^2. \tag{4.1}$$

For $n \ge 4$, one has $c_n < 5/3$, $\widetilde{c}_n < 3$, $d_n < 3$, and $\widetilde{d}_n < 9$.

4.1. **Quadratic and linear majorants.** For the second moment, let $b : [0,1] \to [0,\infty)$ be any convex function such that

$$\beta_s(t) \le b(t) \qquad (0 \le t \le 1).$$

Since $\beta_s(t) = (n-1)^{-1} \sum_{j=1}^{n-1} |1 - atq_j|^2$, we have

$$\frac{1}{n-1} \sum_{j=1}^{n-1} |1 - atq_j|^2 \le b(t). \tag{4.2}$$

Our goal. We need to seek a simple linear function $\ell(t)$ such that

$$\frac{1}{n-1} \sum_{j=1}^{n-1} |1 - atq_j| \le \ell(t), \qquad 0 \le t \le 1. \tag{4.3}$$

Step 1. Estimate the endpoint $t = 1$. By the Cauchy–Schwarz inequality,

$$\left(\frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j| \right)^2 \le \frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j|^2 = \beta_s(1).$$

On the other hand, since $1 - aq_j = -\zeta_j q_j$, the Cauchy–Schwarz inequality gives

$$\begin{aligned} \frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j| &= \frac{1}{n-1} \sum_{j=1}^{n-1} |\zeta_j q_j| \\ &\le \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |\zeta_j|^2 \right)^{1/2} \left(\frac{1}{n-1} \sum_{j=1}^{n-1} |q_j|^2 \right)^{1/2} \\ &= \sqrt{V_* s} \\ &\le \sqrt{V s}. \end{aligned}$$

Therefore

$$\frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j| \le \sqrt{\min\{\beta_s(1), Vs\}}. \tag{4.4}$$

Step 2. Choose the endpoint bound κ. Choose $\kappa \in [0,1]$ such that

$$\kappa^2 \ge \min\{\beta_s(1), Vs\}. \tag{4.5}$$

Then (4.4) implies

$$\frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j| \le \kappa.$$

Define $\ell(t) = 1 - (1-\kappa)t$. Then $\ell(0) = 1$ and $\ell(1) = \kappa$.

Step 3. Interpolate between $t = 0$ and $t = 1$. Since

$$1 - atq_j = (1-t) + t(1 - aq_j),$$

the triangle inequality gives $|1 - atq_j| \le (1-t) + t|1 - aq_j|$. Averaging over j, we obtain

$$\begin{aligned} \frac{1}{n-1} \sum_{j=1}^{n-1} |1 - atq_j| &\le (1-t) + t \frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j| \\ &\le (1-t) + t\kappa \\ &= \ell(t), \end{aligned}$$

which proves (4.3).

In view of (4.2) and (4.3), $b(t)$ controls the averaged second moment of the factors $1 - atq_j$, while $\ell(t)$ controls their averaged moduli.

4.2. **Products with one factor omitted.** In Lemma 4.1, the proof combines two estimates that originate in the arguments of Tao and Zhang. The bound with coefficient 1 is based on the Maclaurin's inequality used in [Tao26, Proposition 11], while the second-moment bound refines the omitted-factor estimate underlying [Zha26b, Lemma 3.2] by retaining the factor s. We shall also use a first-moment analogue and an interval refinement in which the omitted factor is retained quantitatively.

Lemma 4.1. *Let* $\mathcal{A}(t) = \sum_j |q_j|^2 \prod_{k\neq j} |1 - atq_k|$. *Then*

$$\mathcal{A}(t) \le (n-1) \min\left\{\min\{1, c_n s\}\, b(t)^{(n-2)/2},\ \min\{1, \widetilde{c}_n s\}\, \ell(t)^{n-2}\right\}. \tag{4.6}$$

If $I = [l, u] \subset [0, 1]$, $a \le a_+$, *and* $\delta_I = \max\{0, 1 - a_+ u\}$, *then, for all* $t \in I$,

$$\mathcal{A}(t) \le (n-1)s \min\left\{\left(\frac{(n-1)b(t) - \delta_I^2}{n-2}\right)_+^{(n-2)/2},\ \left(\frac{(n-1)\ell(t) - \delta_I}{n-2}\right)_+^{n-2}\right\}. \tag{4.7}$$

Proof. We first prove (4.6). Maclaurin's inequality gives

$$\sum_{j=1}^{n-1} \prod_{k\neq j} |1 - atq_k| \le (n-1)\left(\frac{1}{n-1}\sum_{j=1}^{n-1} |1 - atq_j|\right)^{n-2}.$$

Since $|q_j|^2 \le 1$, it follows from (4.2), (4.3), and the Cauchy–Schwarz inequality that

$$\begin{aligned}\mathcal{A}(t) &\le \sum_{j=1}^{n-1} \prod_{k\neq j} |1 - atq_k| \\ &\le (n-1) \min\left\{b(t)^{(n-2)/2}, \ell(t)^{n-2}\right\}.\end{aligned} \tag{4.8}$$

We now retain the factor $|q_j|^2$. For each fixed j, the arithmetic–geometric mean inequality and (4.2) give

$$\begin{aligned}\prod_{k\neq j} |1 - atq_k| &\le \left(\frac{1}{n-2}\sum_{k\neq j} |1 - atq_k|^2\right)^{(n-2)/2} \\ &\le \left(\frac{n-1}{n-2} b(t)\right)^{(n-2)/2} \\ &= c_n b(t)^{(n-2)/2}.\end{aligned}$$

Similarly, by (4.3),

$$\begin{aligned}\prod_{k\neq j} |1 - atq_k| &\le \left(\frac{1}{n-2}\sum_{k\neq j} |1 - atq_k|\right)^{n-2} \\ &\le \left(\frac{n-1}{n-2} \ell(t)\right)^{n-2} \\ &= \widetilde{c}_n \ell(t)^{n-2}.\end{aligned}$$

Multiplying these two estimates by $|q_j|^2$, summing over j, and using $\sum_{j=1}^{n-1} |q_j|^2 = (n-1)s$, we obtain

$$\mathcal{A}(t) \le (n-1) \min\left\{c_n s\, b(t)^{(n-2)/2}, \widetilde{c}_n s\, \ell(t)^{n-2}\right\}. \tag{4.9}$$

Combining (4.8) and (4.9) proves (4.6).

It remains to prove (4.7). Suppose that $t \in I = [l, u]$. Since $a \le a_+$, $t \le u$, and $|q_j| < 1$, the reverse triangle inequality gives

$$|1 - atq_j| \ge (1 - at|q_j|)_+ \ge (1 - a_+ u)_+ = \delta_I.$$

Hence, for each fixed j,

$$\sum_{k\neq j} |1 - atq_k|^2 \le (n-1)b(t) - \delta_I^2,$$

while

$$\sum_{k\neq j} |1 - atq_k| \leq (n-1)\ell(t) - \delta_I.$$

Applying the arithmetic–geometric mean inequality to the $n-2$ remaining factors therefore gives

$$\prod_{k\neq j} |1 - atq_k| \leq \left(\frac{(n-1)b(t) - \delta_I^2}{n-2}\right)_+^{(n-2)/2}$$

and

$$\prod_{k\neq j} |1 - atq_k| \leq \left(\frac{(n-1)\ell(t) - \delta_I}{n-2}\right)_+^{n-2}.$$

Finally, multiplying by $|q_j|^2$, summing over j, and again using $\sum_{j=1}^{n-1} |q_j|^2 = (n-1)s$, yields (4.7). □

The positive parts in (4.7) allow us to enlarge interval bounds without affecting their validity. On any interval that contains admissible data, the quantities inside the positive parts are already nonnegative. We therefore keep the lower bound δ_I fixed throughout I, rather than replacing it by a t-dependent quantity, which is important when convexity is later used to estimate the corresponding integrals.

4.3. **Centered interpolation.** In the following lemma, the first estimate refines the centered interpolation bound [Zha26b, Lemma 5.1], replacing the single second-moment majorant there by the minimum of the quadratic and linear majorants introduced above. In the present setting, however, $1-a\mu t$ may vanish. We therefore also require a second, nonsingular estimate, obtained by using the second derivative of the interpolation instead of dividing by $1 - a\mu t$.

Lemma 4.2. *For every $t \in [0,1]$ with $1 - a\mu t \neq 0$,*

$$|F(t) - (1 - a\mu t)^{n-1}| \leq \frac{(n-1)a^2t^2v}{2|1 - a\mu t|} \min\{c_n b(t)^{(n-2)/2}, \widetilde{c}_n \ell(t)^{n-2}\}. \tag{4.10}$$

For every $t \in [0,1]$, including zeros of $1 - a\mu t$,

$$|F(t) - (1 - a\mu t)^{n-1}| \leq \frac{(n-1)(n-2)a^2t^2v}{2} \min\{d_n b(t)^{(n-3)/2}, \widetilde{d}_n \ell(t)^{n-3}\}. \tag{4.11}$$

Proof. Step 1. The estimate with denominator. For $0 \leq \theta \leq 1$, define $G_\theta(t) = \prod_{j=1}^{n-1}(1 - a\mu t - at\theta w_j)$. Since $q_j = \mu + w_j$, we have

$$G_0(t) = (1 - a\mu t)^{n-1} \quad \text{and} \quad G_1(t) = F(t).$$

Moreover,

$$1 - a\mu t - at\theta w_j = (1-\theta)(1 - a\mu t) + \theta(1 - atq_j).$$

Using $\sum_{j=1}^{n-1} w_j = 0$, a direct calculation gives

$$\begin{aligned}\frac{1}{n-1}\sum_{j=1}^{n-1} |1 - a\mu t - at\theta w_j|^2 &= |1 - a\mu t|^2 + a^2t^2\theta^2 v \\ &\leq |1 - a\mu t|^2 + a^2t^2 v \\ &= \beta_s(t) \\ &\leq b(t).\end{aligned}$$

For the first absolute moment, the triangle inequality and (4.3) give

$$\begin{aligned}\frac{1}{n-1}\sum_{j=1}^{n-1} |1 - a\mu t - at\theta w_j| &\leq (1-\theta)|1 - a\mu t| + \frac{\theta}{n-1}\sum_{j=1}^{n-1} |1 - atq_j| \\ &\leq \ell(t).\end{aligned}$$

Differentiating $G_\theta(t)$ with respect to θ gives

$$\partial_\theta G_\theta(t) = -at \sum_{j=1}^{n-1} w_j \prod_{k\neq j}(1 - a\mu t - at\theta w_k).$$

Multiplying by $1 - a\mu t$ and using

$$1 - a\mu t = (1 - a\mu t - at\theta w_j) + at\theta w_j$$

together with $\sum_j w_j = 0$, we obtain

$$(1 - a\mu t)\,\partial_\theta G_\theta(t) = -a^2t^2\theta \sum_{j=1}^{n-1} w_j^2 \prod_{k\neq j}(1 - a\mu t - at\theta w_k). \tag{4.12}$$

The same arithmetic–geometric mean argument as in Lemma 4.1, now applied to the factors $|1 - a\mu t - at\theta w_k|$, gives

$$\prod_{k\neq j} |1 - a\mu t - at\theta w_k| \le c_n b(t)^{(n-2)/2}$$

and

$$\prod_{k\neq j} |1 - a\mu t - at\theta w_k| \le \widetilde{c}_n \ell(t)^{n-2}.$$

Since $\sum_{j=1}^{n-1} |w_j|^2 = (n-1)v$, it follows from (4.12) that

$$|\partial_\theta G_\theta(t)| \le \frac{(n-1)a^2t^2\theta v}{|1 - a\mu t|} \min\{c_n b(t)^{(n-2)/2}, \widetilde{c}_n \ell(t)^{n-2}\}.$$

Finally,

$$F(t) - (1 - a\mu t)^{n-1} = G_1(t) - G_0(t) = \int_0^1 \partial_\theta G_\theta(t)\,\mathrm{d}\theta.$$

Integrating the preceding estimate and using $\int_0^1 \theta\,\mathrm{d}\theta = 1/2$ proves (4.10).

Step 2. The estimate without denominator. The argument in Step 1 is not useful when $1 - a\mu t = 0$. To avoid division by this quantity, we instead use the second derivative of the interpolation. Since $\partial_\theta G_0(t) = 0$ by $\sum_j w_j = 0$, Taylor's formula with integral remainder gives

$$G_1(t) - G_0(t) = \int_0^1 (1-\theta)\,\partial_\theta^2 G_\theta(t)\,\mathrm{d}\theta. \tag{4.13}$$

Differentiating once more, we obtain

$$\partial_\theta^2 G_\theta(t) = a^2t^2 \sum_{\substack{j,k=1\\ j\neq k}}^{n-1} w_j w_k \prod_{\substack{r=1\\ r\neq j,k}}^{n-1} (1 - a\mu t - at\theta w_r). \tag{4.14}$$

For the ordered pairs in (4.14), the Cauchy–Schwarz inequality gives

$$\begin{aligned}\sum_{j\neq k} |w_j w_k| &= \left(\sum_{j=1}^{n-1} |w_j|\right)^2 - \sum_{j=1}^{n-1} |w_j|^2 \\ &\le (n-2)\sum_{j=1}^{n-1} |w_j|^2 \\ &= (n-1)(n-2)v.\end{aligned}$$

When two factors are omitted, the arithmetic–geometric mean inequality, together with the bounds established in Step 1, gives

$$\prod_{\substack{r=1\\ r\neq j,k}}^{n-1} |1 - a\mu t - at\theta w_r| \le d_n b(t)^{(n-3)/2}$$

and

$$\prod_{\substack{r=1\\ r\neq j,k}}^{n-1} |1-a\mu t - at\theta w_r| \le \widetilde{d}_n \ell(t)^{n-3}.$$

Consequently,

$$|\partial_\theta^2 G_\theta(t)| \le (n-1)(n-2)a^2t^2v \min\{d_n b(t)^{(n-3)/2}, \widetilde{d}_n \ell(t)^{n-3}\}.$$

Substituting this estimate into (4.13) and using $\int_0^1 (1-\theta)\,\mathrm{d}\theta = 1/2$ gives

$$|G_1(t) - G_0(t)| \le \frac{(n-1)(n-2)a^2t^2v}{2} \min\{d_n b(t)^{(n-3)/2}, \widetilde{d}_n \ell(t)^{n-3}\}.$$

Since $G_1(t) = F(t)$ and $G_0(t) = (1-a\mu t)^{n-1}$, this is precisely (4.11). □

Lemma 4.3. *Suppose that $\mathcal{C}$ is an upper bound for $|a\mu J|$, and let $\mathcal{U} : [0,1] \to [0,\infty)$ satisfy*

$$|F(t) - (1-a\mu t)^{n-1}| \le a^2t^2v\,\mathcal{U}(t), \qquad 0 \le t \le 1.$$

Set

$$K := na^3|\mu| \int_0^1 t^2 \mathcal{U}(t)\,\mathrm{d}t. \tag{4.15}$$

Then every normalized counterexample satisfies

$$1 \le \kappa^{n-1} + \frac{n-1}{n}\mathcal{C} + a^2 \int_0^1 t\,\mathcal{A}(t)\,\mathrm{d}t, \tag{4.16}$$

and

$$1 \le \mathcal{C} + (Vv)^{n/2} + Kv. \tag{4.17}$$

Moreover,

$$1 \le (1+|\mu|)\left\{V^{n/2}(1-|\mu|^2)^{(n-2)/2} + K\right\}. \tag{4.18}$$

Proof. We first prove (4.16). Taking absolute values in (3.5) gives

$$1 \le |F(1)| + (n-1)a|\mu| \left|\int_0^1 F(t)\,\mathrm{d}t\right| + |\mathcal{R}|.$$

By (4.3) at $t=1$ and the arithmetic–geometric mean inequality,

$$\begin{aligned} |F(1)| &= \prod_{j=1}^{n-1} |1-aq_j| \\ &\le \left(\frac{1}{n-1}\sum_{j=1}^{n-1} |1-aq_j|\right)^{n-1} \\ &\le \kappa^{n-1}. \end{aligned} \tag{4.19}$$

Also, (3.4) yields

$$(n-1)a|\mu| \left|\int_0^1 F(t)\,\mathrm{d}t\right| = \frac{n-1}{n}|a\mu J| \le \frac{n-1}{n}\mathcal{C}. \tag{4.20}$$

Finally, by the definition of $\mathcal{R}$,

$$\begin{aligned} |\mathcal{R}| &\le a^2 \int_0^1 t \sum_{j=1}^{n-1} |q_j|^2 \prod_{k\neq j} |1-atq_k|\,\mathrm{d}t \\ &= a^2 \int_0^1 t\,\mathcal{A}(t)\,\mathrm{d}t. \end{aligned} \tag{4.21}$$

Combining (4.19), (4.20), and (4.21) proves (4.16).

We next derive (4.17). Taking absolute values in (3.6), we obtain

$$1 \le |a\mu J| + |1-a\mu|^n + na|\mu| \int_0^1 |F(t) - (1-a\mu t)^{n-1}|\,\mathrm{d}t.$$

By (3.2), $|1-a\mu|^2 \le Vv$, and by the defining property of $\mathcal{U}$,

$$na|\mu| \int_0^1 |F(t) - (1-a\mu t)^{n-1}| \, \mathrm{d}t \le Kv.$$

Since $|a\mu J| \le \mathcal{C}$, it follows that

$$1 \le \mathcal{C} + (Vv)^{n/2} + Kv,$$

which is (4.17).

It remains to prove (4.18). From (3.7), together with $\Phi_{n-1}(a) \le 1$ and $|q_j| < 1$, we have

$$|a\mu J| \le |\mu|.$$

Using this sharper bound directly in (3.6) gives

$$1 - |\mu| \le (Vv)^{n/2} + Kv. \tag{4.22}$$

Since $v = s - |\mu|^2 \le 1 - |\mu|^2$, the right-hand side of (4.22) satisfies

$$\begin{aligned}(Vv)^{n/2} + Kv &= v\left\{V^{n/2} v^{(n-2)/2} + K\right\} \\ &\le (1-|\mu|^2)\left\{V^{n/2}(1-|\mu|^2)^{(n-2)/2} + K\right\}.\end{aligned}$$

Moreover, $|\mu| < 1$, since $|q_j| < 1$ for every j. Hence $1 - |\mu| > 0$, and dividing by $1 - |\mu|$ gives

$$1 \le (1+|\mu|)\left\{V^{n/2}(1-|\mu|^2)^{(n-2)/2} + K\right\},$$

which proves (4.18). □

Remark 4.4. The derivation of (4.18) uses the actual bound $|a\mu J| \le |\mu|$ before any interval enlargement is made. An independently chosen interval upper bound $\mathcal{C}$ need not satisfy $\mathcal{C} \le |\mu|$, and therefore cannot in general be substituted for $|a\mu J|$ in this argument.

4.4. **Convex integration on a rational mesh.** Let f be a nonnegative convex function on $[l, u]$. By convexity, f lies below the affine chord joining its endpoint values; that is,

$$f(t) \le \frac{u-t}{u-l} f(l) + \frac{t-l}{u-l} f(u), \qquad l \le t \le u.$$

Multiplying this inequality by t and t^2, respectively, and integrating over $[l, u]$, we obtain

$$\int_l^u t f(t) \, \mathrm{d}t \le \frac{(u-l)(2l+u)}{6} f(l) + \frac{(u-l)(l+2u)}{6} f(u), \tag{4.23}$$

$$\int_l^u t^2 f(t) \, \mathrm{d}t \le \frac{(u-l)(3l^2+2lu+u^2)}{12} f(l) + \frac{(u-l)(l^2+2lu+3u^2)}{12} f(u). \tag{4.24}$$

All four coefficients on the right-hand sides are nonnegative for $0 \le l \le u \le 1$.

We now explain how these bounds are applied to the majorants introduced above. Since b is nonnegative and convex, b^α is convex whenever $\alpha \ge 1$. Likewise, ℓ is nonnegative and affine, so ℓ^α is convex for every $\alpha \ge 1$. Consequently, each of the individual majorants occurring in (4.6) is convex for $n \ge 4$. The same conclusion holds for the retained bounds in (4.7): taking a positive part preserves convexity, and composing a nonnegative convex function with $x \mapsto x^\alpha$, $\alpha \ge 1$, again gives a convex function.

For the centered interpolation estimates, the corresponding powers are also convex when $n \ge 5$. In particular, the exponents

$$\frac{n-2}{2}, \qquad n-2, \qquad \frac{n-3}{2}, \qquad n-3$$

are then all at least one. On a mesh interval $I = [l, u]$ for which $|1 - a\mu t|$ has a positive constant lower bound, we replace the denominator in (4.10) by that constant lower bound and apply (4.24) to each resulting convex majorant. If no positive lower bound for $|1 - a\mu t|$ is available on I, we use only the nonsingular estimate (4.11).

Whenever several convex majorants are available, we integrate each one separately using (4.23) or (4.24), and only afterwards take the smallest of the resulting upper bounds. This order is important: although each individual majorant is convex, their pointwise minimum need not be convex, so the chord inequalities cannot in general be applied directly to that minimum.

For $n = 4$, the exponent $(n-3)/2 = 1/2$ occurring in (4.11) is below one, and the preceding convexity argument no longer applies to that branch. We therefore do not use the centered integral tests in degree 4.

5. The analytic argument for large degrees

The covariance and integral estimates developed above are already strong enough to exclude all sufficiently large degrees, without any finite verification. We use the cutoff $n = 100001$; the estimates below leave a uniform margin in each of the four ranges of a.

Proposition 5.1. *There is no normalized counterexample with $n \geq 100001$.*

Proof. Throughout the proof we use $V = 1$. From (3.1), $|\mu - a|^2 \leq a^2(1-s)$, and hence

$$x \geq a - |a - \mu| \geq a(1 - \sqrt{1-s}).$$

On the other hand, (3.2) gives $s(1+a^2) \geq 1$, so that

$$1 - s \leq \frac{a^2}{1+a^2}.$$

Consequently,

$$x \geq a(1 - \sqrt{1-s}) \geq a\left(1 - \frac{a}{\sqrt{1+a^2}}\right) \geq \frac{a}{2(1+a^2)}. \tag{5.1}$$

For the last inequality, rationalization gives

$$1 - \frac{a}{\sqrt{1+a^2}} = \frac{1}{\sqrt{1+a^2}(\sqrt{1+a^2}+a)} \geq \frac{1}{2(1+a^2)}.$$

Set $g(a) = a^2/[4(1+a^2)^2]$. Since $\rho \geq |\mu|^2 \geq x^2$, (5.1) and the definition of β_s imply

$$\beta_s(t) = 1 - \rho t - a^2 s\, t(1-t) \leq 1 - g(a)t.$$

We first obtain a consequence that will be used repeatedly. Suppose more generally that

$$\beta_s(t) \leq 1 - gt, \qquad 0 \leq t \leq 1,$$

for some $g > 0$. By the arithmetic–geometric mean inequality,

$$\begin{aligned}
|F(1)| &= \prod_{j=1}^{n-1} |1 - aq_j| \\
&\leq \left(\frac{1}{n-1}\sum_{j=1}^{n-1} |1 - aq_j|^2\right)^{(n-1)/2} \\
&= \beta_s(1)^{(n-1)/2} \\
&\leq (1-g)^{(n-1)/2} \\
&\leq e^{-(n-1)g/2}.
\end{aligned}$$

Likewise, Lemma 4.1, with $b = \beta_s$, gives

$$\mathcal{A}(t) \leq (n-1)\beta_s(t)^{(n-2)/2} \leq (n-1)e^{-(n-2)gt/2}.$$

It follows from the definition of $\mathcal{R}$ that

$$\begin{aligned}|\mathcal{R}| &\le a^2(n-1)\int_0^1 t\,e^{-(n-2)gt/2}\,\mathrm{d}t\\ &\le a^2(n-1)\int_0^\infty t\,e^{-(n-2)gt/2}\,\mathrm{d}t\\ &= \frac{4(n-1)a^2}{(n-2)^2g^2}.\end{aligned}$$

Thus

$$|F(1)| \le e^{-(n-1)g/2}, \qquad |\mathcal{R}| \le \frac{4(n-1)a^2}{(n-2)^2g^2}. \tag{5.2}$$

Taking absolute values in (3.5), and then using (3.4) and (3.7), gives

$$\begin{aligned}1 &\le |F(1)| + \frac{n-1}{n}|a\mu J| + |\mathcal{R}|\\ &\le |F(1)| + \psi(a) + |\mathcal{R}|.\end{aligned}$$

Hence every normalized counterexample must satisfy

$$1 \le |F(1)| + \psi(a) + |\mathcal{R}|. \tag{5.3}$$

We now derive a contradiction in four ranges of a.

Step 1. $0 < a \le 1/10$. Since $|a-\zeta_j| > 1$ for every j,

$$|P'(a)| = n\prod_{j=1}^{n-1}|a-\zeta_j| > n.$$

On the other hand, $P'(a) = \prod_{j=1}^{n-1}(a-z_j)$, and, as in the proof of Lemma 3.3,

$$\frac{1}{n-1}\sum_{j=1}^{n-1}|a-z_j|^2 = \frac{n(1+a^2)}{n-1}.$$

The arithmetic–geometric mean inequality therefore yields

$$n < \left(\frac{n(1+a^2)}{n-1}\right)^{(n-1)/2}.$$

Taking logarithms gives

$$(n-1)\log(1+a^2) > 2\log n - (n-1)\log\left(1+\frac{1}{n-1}\right).$$

Since $(n-1)\log\left(1+\frac{1}{n-1}\right) < 1$ and $\log(1+a^2) < a^2$, we obtain

$$(n-1)a^2 > 2\log n - 1 > 19. \tag{5.4}$$

For $a \le 1/10$, the middle expression in (5.1) satisfies

$$a\left(1-\frac{a}{\sqrt{1+a^2}}\right) \ge \frac{9a}{10}.$$

Hence

$$\rho \ge x^2 \ge \frac{81}{100}a^2 > \frac{4}{5}a^2,$$

and therefore

$$\beta_s(t) \le 1 - \frac{4}{5}a^2t.$$

Applying (5.2) with $g = \frac{4}{5}a^2$, and using (5.4), gives

$$|F(1)| \le e^{-\frac{2}{5}(n-1)a^2} < \frac{1}{100}.$$

Moreover,

$$|\mathcal{R}| \le \frac{25(n-1)}{4(n-2)^2a^2} < \frac{8}{(n-1)a^2} < \frac{8}{19}.$$

Finally, ψ is increasing on $(0,1)$, and hence

$$\psi(a) \le \psi(1/10) < \frac{1}{6}.$$

Substitution into (5.3) gives

$$1 < \frac{1}{100} + \frac{1}{6} + \frac{8}{19} < 1,$$

a contradiction.

Step 2. $a \in [1/10, 3/4] \cup [5/4, 2]$. The function $g(a) = \frac{a^2}{4(1+a^2)^2}$ is increasing on $(0,1]$ and decreasing on $[1,\infty)$. Hence, on the present range, $g(a) \ge \frac{1}{500}$. Equation (5.2) therefore gives

$$|F(1)| \le e^{-(n-1)/1000} < \frac{1}{1000}.$$

Set $H_0(a) = \frac{16(1+a^2)^4}{a^2}$. Since $g = g(a)$, the second estimate in (5.2) becomes

$$|\mathcal{R}| \le \frac{4H_0(a)(n-1)}{(n-2)^2} < \frac{5H_0(a)}{n-1}.$$

The required bounds for ψ and H_0 are summarized in Table 1.

TABLE 1. Bounds away from $a = 1$ in the large-degree argument.

Range of a	Upper bound for $\psi(a)$	Upper bound for $H_0(a)$
$[1/10, 1/2]$	$3/4$	1665
$[1/2, 3/4]$	$19/20$	170
$[5/4, 3/2]$	$19/20$	794
$[3/2, 2]$	$17/20$	2500

To verify the table, recall that ψ increases on $(0,1]$ and decreases on $[1,\infty)$. If $u = a^2$, then

$$H_0(a) = 16\frac{(1+u)^4}{u},$$

whose logarithmic derivative has the sign of $3u - 1$. Thus only the appropriate endpoints of the four intervals need to be checked. The bounds for ψ follow, for example, from

$$e^{3/8} < \frac{3}{2}, \qquad e^{7/32} < \frac{19}{15}, \qquad e^{9/32} > \frac{25}{19}, \qquad e^{5/8} > \frac{30}{17}.$$

These elementary exponential comparisons may be verified directly from the exponential series; an exact rational check is included in the supplement.

Since $n - 1 \ge 100000$, the right-hand side of (5.3) is, in each row of the table, strictly less than one. The largest of the four resulting upper bounds is

$$\frac{1}{1000} + \frac{19}{20} + \frac{5 \cdot 794}{100000} = \frac{9907}{10000} < 1.$$

This again contradicts (5.3).

Step 3. $3/4 \le a \le 5/4$. From (5.1),

$$|\mu| \ge x \ge \frac{6}{25}.$$

Set $B_* = \frac{589}{625}$. Since $\rho \ge |\mu|^2$, we have

$$\rho \ge \frac{36}{625}, \qquad \beta_s(1) = 1 - \rho \le B_*.$$

Also, (3.2) gives

$$s(1+a^2) \geq 1,$$

and hence, since $a \geq 3/4$,

$$a^2 s \geq \frac{a^2}{1+a^2} \geq \frac{9}{25}.$$

For $0 \leq t \leq 1/2$,

$$\begin{aligned}\beta_s(t) &= 1 - \rho t - a^2 s\, t(1-t) \\ &\leq 1 - t\left(\frac{36}{625} + \frac{9}{25}(1-t)\right) \\ &\leq 1 - \frac{t}{5}.\end{aligned}$$

Moreover,

$$\beta_s(1/2) \leq 1 - \frac{18}{625} - \frac{9}{100} = \frac{2203}{2500} < \frac{9}{10}.$$

Since β_s is convex, its maximum on $[1/2, 1]$ is attained at an endpoint. Together with $\beta_s(1) \leq B_*$, this gives

$$\beta_s(t) \leq 1 - t/5 \quad (0 \leq t \leq 1/2), \qquad \beta_s(t) \leq B_* \quad (1/2 \leq t \leq 1). \tag{5.5}$$

Finally, for $0 \leq t \leq 1/2$,

$$|1 - a\mu t| \geq 1 - a|\mu| t \geq 1 - \frac{5}{8} = \frac{3}{8}.$$

Let

$$E = n|a\mu| \int_0^1 |F(t) - (1-a\mu t)^{n-1}|\, \mathrm{d}t,$$

and write $E = E_1 + E_2$, where the two terms correspond to $[0, 1/2]$ and $[1/2, 1]$, respectively.

For E_1, use (4.10), (5.5), $c_n < 5/3$, $a \leq 5/4$, $|a\mu| \leq 5/4$, and $|1 - a\mu t| \geq 3/8$. Since

$$\left(1 - \frac{t}{5}\right)^{(n-2)/2} \leq e^{-(n-2)t/10},$$

we obtain

$$\begin{aligned}E_1 &\leq \frac{625}{144} n(n-1) v \int_0^\infty t^2 e^{-(n-2)t/10}\, \mathrm{d}t \\ &= \frac{78125}{9} \frac{n(n-1)}{(n-2)^3} v \leq \frac{10000}{n-1} v \leq \frac{v}{10}.\end{aligned} \tag{5.6}$$

For E_2, we use the nonsingular estimate (4.11), together with $d_n < 3$ and the second estimate in (5.5). Since $\int_{1/2}^1 t^2\, \mathrm{d}t = \frac{7}{24}$, we obtain

$$\begin{aligned}E_2 &\leq 2n(n-1)^2 v B_*^{(n-3)/2} \\ &\leq 2n(n-1)^2 v\, e^{-(n-1)/40} \\ &\leq \frac{6 \cdot 10^{12}}{(n-1)^3} v < \frac{v}{10},\end{aligned} \tag{5.7}$$

where for the second inequality, note that

$$-\log B_* = -\log\left(1 - \frac{36}{625}\right) > \frac{36}{625} > \frac{1}{18},$$

and hence, for $n \geq 100001$,

$$B_*^{(n-3)/2} \leq e^{-(n-1)/40}.$$

For the third inequality, we use

$$e^{(n-1)/40} \geq \frac{((n-1)/40)^6}{720}.$$

It remains to bound the endpoint term in (3.6). Since $v \le 1-|\mu|^2 \le B_*$, we have

$$\begin{aligned} v^{n/2} &= v\, v^{(n-2)/2} \\ &\le (1-|\mu|^2)B_*^{(n-2)/2} \\ &\le 2(1-|\mu|)B_*^{(n-2)/2} \\ &\le \frac{1-|\mu|}{5}. \end{aligned}$$

Now (3.6), (3.2), (3.7), (5.6), and (5.7) give

$$\begin{aligned} 1 &\le |\mu| + v^{n/2} + E_1 + E_2 \\ &\le |\mu| + \frac{1-|\mu|}{5} + \frac{v}{5} \\ &\le |\mu| + \frac{3}{5}(1-|\mu|) \\ &< 1, \end{aligned}$$

where in the last two lines we used $v \le 1-|\mu|^2 \le 2(1-|\mu|)$ and $|\mu| < 1$. This is the desired contradiction.

Step 4. $a \ge 2$. From (3.1), with $V = 1$,

$$|\mu - a|^2 \le a^2(1-s).$$

Expanding the left-hand side gives

$$a^2 s \le 2ax - |\mu|^2 \le 2a|\mu| - |\mu|^2 \le 2a - 1.$$

Since the function $(2a-1)/a^2$ is decreasing for $a \ge 2$, it follows that

$$s \le \frac{2a-1}{a^2} \le \frac{3}{4}.$$

We also have

$$\beta_s(1) = \frac{1}{n-1}\sum_{j=1}^{n-1} |1 - aq_j|^2 \le 1,$$

because $\beta_s(1) = 1-\rho$ and $\rho \ge 0$. Minkowski's inequality therefore gives

$$\begin{aligned} a\sqrt{s} &= \left(\frac{1}{n-1}\sum_{j=1}^{n-1} |aq_j|^2\right)^{1/2} \\ &\le 1 + \left(\frac{1}{n-1}\sum_{j=1}^{n-1} |1-aq_j|^2\right)^{1/2} \\ &= 1 + \sqrt{\beta_s(1)} \le 2. \end{aligned}$$

Thus $a^2 s \le 4$.

Moreover, since $1 - aq_j = -\zeta_j q_j$, the Cauchy–Schwarz inequality gives

$$\begin{aligned} \frac{1}{n-1}\sum_{j=1}^{n-1} |1-aq_j| &= \frac{1}{n-1}\sum_{j=1}^{n-1} |\zeta_j q_j| \\ &\le \sqrt{V_* s} \\ &\le \sqrt{s} \le \frac{\sqrt{3}}{2} < \frac{7}{8}. \end{aligned}$$

We may therefore take

$$\ell(t) = 1 - \frac{t}{8}.$$

Using the linear branch of Lemma 4.1, $\widetilde{c}_n < 3$, and $a^2 s \le 4$, we obtain

$$\begin{aligned}|\mathcal{R}| &\le a^2(n-1)\widetilde{c}_n s \int_0^1 t\left(1-\frac{t}{8}\right)^{n-2}\,\mathrm{d}t \\ &\le 12(n-1)\int_0^8 t\left(1-\frac{t}{8}\right)^{n-2}\,\mathrm{d}t \\ &= \frac{768}{n}.\end{aligned}$$

Similarly,

$$|F(1)| \le \left(\frac{7}{8}\right)^{n-1}.$$

Finally, ψ decreases on $[1,\infty)$, so

$$\psi(a) \le \psi(2) = 2e^{-3/2} < \frac{9}{20}.$$

Substituting these bounds into (5.3) gives

$$1 \le \left(\frac{7}{8}\right)^{n-1} + \frac{9}{20} + \frac{768}{n} < 1,$$

since $n \ge 100001$. This final contradiction completes the proof. □

6. Product bounds for the small degrees

The direct estimates become less effective when both the reciprocal mean and the relevant products are close to their extremal values. To obtain the additional information needed in the small-degree range, we shall retain more of the covariance structure of the original root distances and of the variation in the reciprocal moduli. The notation introduced in Section 2 remains in force.

6.1. An elementary symmetric inequality.

Recall the elementary symmetric polynomials e_k introduced in Section 2. The following inequality provides the refinement of the arithmetic–geometric mean estimate that will be used in Lemma 6.3.

Lemma 6.1. *Let $n \ge 4$, and let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1})$ satisfy $\xi_j \ge 0$ for all $1 \le j \le n-1$ and $\sum_{j=1}^{n-1} \xi_j = n-1$. Then*

$$e_{n-2}(\boldsymbol{\xi}) \le 3 + (n-4)e_{n-1}(\boldsymbol{\xi}). \tag{6.1}$$

Proof. Consider $f(\boldsymbol{\xi}) = e_{n-2}(\boldsymbol{\xi}) - (n-4)e_{n-1}(\boldsymbol{\xi})$ on the compact simplex

$$\left\{\boldsymbol{\xi} \in [0,\infty)^{n-1} : \sum_{j=1}^{n-1} \xi_j = n-1\right\}.$$

It is enough to show that the maximum of f is 3.

Step 1. Boundary points. Suppose first that $\boldsymbol{\xi}$ lies on the boundary of the simplex. If at least two coordinates vanish, then every product appearing in $e_{n-2}(\boldsymbol{\xi})$ contains a zero, and hence

$$e_{n-2}(\boldsymbol{\xi}) = e_{n-1}(\boldsymbol{\xi}) = 0.$$

Thus $f(\boldsymbol{\xi}) = 0$.

If exactly one coordinate vanishes, say $\xi_{n-1} = 0$, then

$$e_{n-1}(\boldsymbol{\xi}) = 0, \qquad e_{n-2}(\boldsymbol{\xi}) = \prod_{j=1}^{n-2} \xi_j.$$

Since $\sum_{j=1}^{n-2} \xi_j = n-1$, the arithmetic–geometric mean inequality gives

$$f(\boldsymbol{\xi}) = \prod_{j=1}^{n-2} \xi_j \le \left(\frac{n-1}{n-2}\right)^{n-2} < 3.$$

On the other hand, at $\boldsymbol{\xi} = (1, \dots, 1)$ we have

$$e_{n-2}(\boldsymbol{\xi}) = n - 1, \qquad e_{n-1}(\boldsymbol{\xi}) = 1,$$

and therefore

$$f(1, \dots, 1) = 3.$$

It remains only to exclude a nonconstant interior maximizer with value larger than 3.

Step 2. Interior stationary points. Let $\boldsymbol{\xi}$ be an interior maximizer. Since the constraint is $\sum_{j=1}^{n-1} \xi_j = n - 1$, the Lagrange multiplier equations imply

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial f}{\partial \xi_j} \qquad (1 \le i, j \le n-1).$$

For $i \neq j$, a direct calculation gives

$$\frac{\partial f}{\partial \xi_i} - \frac{\partial f}{\partial \xi_j} = (\xi_j - \xi_i) \prod_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \xi_\ell \left(\sum_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \frac{1}{\xi_\ell} - (n-4) \right). \tag{6.2}$$

Since all coordinates are positive, whenever $\xi_i \neq \xi_j$, (6.2) yields

$$\sum_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \frac{1}{\xi_\ell} = n - 4. \tag{6.3}$$

We claim that at most two distinct coordinate values can occur. Indeed, suppose that ξ_i, ξ_j, ξ_k are three distinct values. Applying (6.3) to the pairs (i, j) and (i, k) and subtracting the resulting identities gives

$$\frac{1}{\xi_k} = \frac{1}{\xi_j},$$

and hence $\xi_j = \xi_k$, a contradiction. Thus every nonconstant interior stationary point has exactly two distinct coordinate values, say $u \neq w$.

Step 3. Exclusion of nonconstant interior maxima. We first show that u and w cannot both occur at least twice. Suppose that $\xi_i = \xi_j = u$. Along the tangent direction in which ξ_i is replaced by $u + t$ and ξ_j by $u - t$, while all other coordinates are held fixed, the second derivative at $t = 0$ is

$$-2 \prod_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \xi_\ell \left(\sum_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \frac{1}{\xi_\ell} - (n-4) \right). \tag{6.4}$$

At a local maximum this quantity must be nonpositive. Since all coordinates are positive, (6.4) therefore implies

$$\sum_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \frac{1}{\xi_\ell} \ge n - 4.$$

Now choose a mixed pair consisting of one coordinate equal to u and one coordinate equal to w. By (6.3),

$$\sum_{\substack{1 \le \ell \le n-1 \\ \ell \neq i,j}} \frac{1}{\xi_\ell} = n - 4$$

for such a pair. Comparing this identity with the preceding inequality for a pair u, u gives

$$\frac{1}{w} - \frac{1}{u} \ge 0,$$

and hence $u \ge w$. If w also occurs at least twice, the same argument applied to a pair w, w gives

$$\frac{1}{u} - \frac{1}{w} \ge 0,$$

and hence $w \geq u$. This contradicts $u \neq w$.

Thus one of the two values occurs exactly once. Relabeling if necessary, suppose that u occurs $n-2$ times and w occurs once. Applying (6.3) to a mixed pair u, w leaves $n-3$ coordinates, all equal to u, and therefore

$$\frac{n-3}{u} = n-4. \tag{6.5}$$

For $n = 4$, (6.5) is impossible. For $n > 4$, it gives

$$u = \frac{n-3}{n-4}.$$

Using $(n-2)u + w = n-1$, we then obtain

$$w = n-1-\frac{(n-2)(n-3)}{n-4} = -\frac{2}{n-4} < 0,$$

contrary to the positivity of the coordinates.

Hence there is no nonconstant interior maximum. The only remaining interior maximizer is $\boldsymbol{\xi} = (1, \ldots, 1)$, where $f = 3$. Together with Step 1, this proves (6.1). □

6.2. **The original root distances.** Define

$$\begin{gathered} A_0 = \frac{1}{n-1}\sum_{j=1}^{n-1}(a-z_j) = \frac{na}{n-1}, \qquad B_0 = \frac{1}{n-1}\sum_{j=1}^{n-1}|a-z_j|^2 = \frac{n(1+a^2)}{n-1}, \\ C_0 = B_0 - A_0^2 = \frac{n(n-1-a^2)}{(n-1)^2}. \end{gathered} \tag{6.6}$$

The logarithmic derivative identity of Meir and Sharma [MS69, equation (3.2)] relates these root distances to the reciprocal critical-point mean. Tao [Tao26, Lemma 15] states its simple-zero case explicitly.

Lemma 6.2 ([MS69, equation (3.2)])**.** *For the original root distances $a - z_j$ in (6.6), one has*

$$\frac{1}{n-1}\sum_{j=1}^{n-1}\frac{1}{a-z_j} = \frac{\mu}{2}.$$

Lemma 6.3. *If $C_0 > 0$, then*

$$\prod_{j=1}^{n-1}|q_j|^2 \geq \frac{n^2}{B_0^{n-1}}(1+\gamma), \qquad \text{where } \gamma := \frac{n-1}{3}\frac{|A_0 - B_0\mu/2|^2}{C_0}. \tag{6.7}$$

If $C_0 = 0$, then $A_0 - B_0\mu/2 = 0$, and the same bound holds with $\gamma = 0$.

Proof. Assume first that $C_0 > 0$. Set

$$T = \frac{1}{n-1}\sum_{j=1}^{n-1}\left|\frac{1}{a-z_j}\right|^2, \qquad M = \frac{1}{n-1}\sum_{j=1}^{n-1}\frac{1}{a-z_j} = \frac{\mu}{2},$$

where the second identity follows from Lemma 6.2.

By the definitions of A_0, B_0, C_0,

$$\frac{1}{n-1}\sum_{j=1}^{n-1}|(a-z_j) - A_0|^2 = C_0,$$

while

$$\frac{1}{n-1}\sum_{j=1}^{n-1}\left|\frac{1}{a-z_j} - M\right|^2 = T - |M|^2.$$

Moreover,

$$\frac{1}{n-1}\sum_{j=1}^{n-1}((a-z_j)-A_0)\left(\frac{1}{a-z_j}-M\right)=1-A_0M.$$

Hence the Cauchy–Schwarz inequality gives

$$|1-A_0M|^2\le C_0(T-|M|^2). \tag{6.8}$$

Expanding (6.8), and using $C_0=B_0-A_0^2$, we obtain

$$C_0T\ge 1-2A_0\Re M+B_0|M|^2.$$

Multiplying by B_0 and observing that

$$C_0+|A_0-B_0M|^2=B_0(1-2A_0\Re M+B_0|M|^2),$$

we obtain

$$B_0T\ge 1+\frac{|A_0-B_0M|^2}{C_0}. \tag{6.9}$$

We now apply Lemma 6.1 to

$$\xi_j=\frac{|a-z_j|^2}{B_0},\qquad 1\le j\le n-1.$$

By the definition of B_0,

$$\sum_{j=1}^{n-1}\xi_j=n-1.$$

Since every $\xi_j>0$,

$$\begin{aligned}e_{n-2}(\boldsymbol{\xi})&=e_{n-1}(\boldsymbol{\xi})\sum_{j=1}^{n-1}\frac{1}{\xi_j}\\&=e_{n-1}(\boldsymbol{\xi})B_0\sum_{j=1}^{n-1}\frac{1}{|a-z_j|^2}\\&=e_{n-1}(\boldsymbol{\xi})(n-1)B_0T.\end{aligned}$$

Therefore (6.1) gives

$$e_{n-1}(\boldsymbol{\xi})\big((n-1)B_0T-(n-4)\big)\le 3.$$

Using (6.9), we have

$$\begin{aligned}(n-1)B_0T-(n-4)&\ge(n-1)\left(1+\frac{|A_0-B_0M|^2}{C_0}\right)-(n-4)\\&=3+(n-1)\frac{|A_0-B_0M|^2}{C_0}\\&=3(1+\gamma),\end{aligned}$$

because $M=\mu/2$. Consequently,

$$e_{n-1}(\boldsymbol{\xi})(1+\gamma)\le 1. \tag{6.10}$$

Finally,

$$e_{n-1}(\boldsymbol{\xi})=\frac{\prod_{j=1}^{n-1}|a-z_j|^2}{B_0^{n-1}}.$$

Since $P'(a)=\prod_{j=1}^{n-1}(a-z_j)=n\prod_{j=1}^{n-1}(a-\zeta_j)$, and $q_j=(a-\zeta_j)^{-1}$, we obtain

$$\prod_{j=1}^{n-1}|a-z_j|^2=|P'(a)|^2=\frac{n^2}{\prod_{j=1}^{n-1}|q_j|^2}.$$

Thus

$$e_{n-1}(\boldsymbol{\xi}) = \frac{n^2}{B_0^{n-1}\prod_{j=1}^{n-1}|q_j|^2}.$$

Substituting this identity into (6.10) gives (6.7).

It remains to consider $C_0 = 0$. In this case,

$$\frac{1}{n-1}\sum_{j=1}^{n-1}|(a-z_j)-A_0|^2 = C_0 = 0,$$

so

$$a - z_j = A_0 \qquad (1 \le j \le n-1).$$

Hence Lemma 6.2 gives

$$M = \frac{1}{A_0} = \frac{\mu}{2}.$$

Since $B_0 = A_0^2$, it follows that

$$A_0 - B_0\frac{\mu}{2} = A_0 - B_0 M = 0.$$

Moreover,

$$B_0^{n-1} = \prod_{j=1}^{n-1}|a-z_j|^2 = \frac{n^2}{\prod_{j=1}^{n-1}|q_j|^2},$$

so (6.7) holds, in fact with equality, when $\gamma = 0$. □

6.3. **Variation of the reciprocal moduli.**

Lemma 6.4. *If $V > 0$, then*

$$\prod_{j=1}^{n-1}|q_j|^2 \le s^{n-1}\exp\left(-\frac{(n-1)|\mu - as|^2}{2sV}\right). \tag{6.11}$$

For $V = 0$, the bound $\prod_{j=1}^{n-1}|q_j|^2 \le s^{n-1}$ remains valid.

Proof. We begin with the elementary inequality

$$\log u \le \log s + \frac{u-s}{s} - \frac{(u-s)^2}{2s}, \qquad 0 < u, s \le 1. \tag{6.12}$$

Indeed, let

$$h(u) = \log u - \log s - \frac{u-s}{s} + \frac{(u-s)^2}{2s}.$$

Then $h(s) = 0$, while

$$h'(u) = \frac{(u-1)(u-s)}{su}.$$

If $0 < u < s$, then $h'(u) \ge 0$ on $(0, s)$, and hence $h(u) \le h(s) = 0$. If $s < u \le 1$, then $h'(u) \le 0$ on $(s, 1]$, and again $h(u) \le h(s) = 0$. This proves (6.12).

We now apply (6.12) with $u = |q_j|^2$ and sum over $1 \le j \le n-1$. Since $\sum_{j=1}^{n-1}(|q_j|^2 - s) = 0$ by the definition of s, the linear terms cancel, and we obtain

$$\log\prod_{j=1}^{n-1}|q_j|^2 \le (n-1)\log s - \frac{1}{2s}\sum_{j=1}^{n-1}(|q_j|^2 - s)^2. \tag{6.13}$$

It remains to estimate the last sum from below. Since

$$\mu - as = -\frac{1}{n-1}\sum_{j=1}^{n-1}\overline{\zeta_j}(|q_j|^2 - s),$$

the Cauchy–Schwarz inequality gives

$$\begin{aligned}|\mu - as|^2 &\le \left(\frac{1}{n-1}\sum_{j=1}^{n-1}|\zeta_j|^2\right)\left(\frac{1}{n-1}\sum_{j=1}^{n-1}(|q_j|^2 - s)^2\right)\\ &= V_*\left(\frac{1}{n-1}\sum_{j=1}^{n-1}(|q_j|^2 - s)^2\right)\\ &\le V\left(\frac{1}{n-1}\sum_{j=1}^{n-1}(|q_j|^2 - s)^2\right).\end{aligned}$$

Thus, when $V > 0$,

$$\sum_{j=1}^{n-1}(|q_j|^2 - s)^2 \ge \frac{(n-1)|\mu - as|^2}{V}. \tag{6.14}$$

Substituting (6.14) into (6.13) yields

$$\log\prod_{j=1}^{n-1}|q_j|^2 \le (n-1)\log s - \frac{(n-1)|\mu - as|^2}{2sV}.$$

Exponentiating gives (6.11).

If $V = 0$, no division by V is needed. The arithmetic–geometric mean inequality and the definition of s give directly

$$\prod_{j=1}^{n-1}|q_j|^2 \le \left(\frac{1}{n-1}\sum_{j=1}^{n-1}|q_j|^2\right)^{n-1} = s^{n-1},$$

as claimed. □

We next use the following product inequality from Tao's exposition. Its nonvanishing hypothesis is automatic for reciprocal critical-point coordinates.

Lemma 6.5 ([Tao26, Lemma 13])**.** *If $0 < |\xi_j| \le 1$ for $1 \le j \le N$, then*

$$\left(\prod_{j=1}^{N}|\xi_j|\right)\sum_{j=1}^{N}\left|\frac{1}{\xi_j} - \overline{\xi_j}\right| \le 1 - \prod_{j=1}^{N}|\xi_j|^2.$$

This inequality and the reciprocal second moment give two bounds on $|a - \mu|$.

Lemma 6.6. *If $(n-1)(1-s) < 1$, then*

$$|a-\mu|^2 \le \frac{(1-s)^2}{1-(n-1)(1-s)}, \qquad V_* \le 1 + \frac{1-s}{1-(n-1)(1-s)} - a^2. \tag{6.15}$$

Without this additional restriction,

$$(n-1)|a-\mu| \le \left(\prod_{j=1}^{n-1}|q_j|^2\right)^{-1/2} - \left(\prod_{j=1}^{n-1}|q_j|^2\right)^{1/2}. \tag{6.16}$$

Proof. Assume first that $(n-1)(1-s) < 1$. Since $\sum_{j=1}^{n-1}(1-|q_j|^2) = (n-1)(1-s)$ and every summand is nonnegative, we have, for each j,

$$|q_j|^2 \ge 1 - (n-1)(1-s) > 0. \tag{6.17}$$

Moreover, since $1/q_j = a - \zeta_j$ and $\sum_{j=1}^{n-1}\zeta_j = 0$, we have

$$\frac{1}{n-1}\sum_{j=1}^{n-1}\frac{1}{q_j} = a.$$

Therefore

$$\begin{aligned}|a-\mu| &= |a-\overline{\mu}| \\ &= \left|\frac{1}{n-1}\sum_{j=1}^{n-1}\left(\frac{1}{q_j}-\overline{q_j}\right)\right| \\ &\le \frac{1}{n-1}\sum_{j=1}^{n-1}\left|\frac{1}{q_j}-\overline{q_j}\right| \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1}\frac{1-|q_j|^2}{|q_j|}.\end{aligned}$$

Using (6.17), we obtain

$$\begin{aligned}|a-\mu| &\le \frac{1}{\sqrt{1-(n-1)(1-s)}}\frac{1}{n-1}\sum_{j=1}^{n-1}(1-|q_j|^2) \\ &= \frac{1-s}{\sqrt{1-(n-1)(1-s)}}.\end{aligned}$$

Squaring gives the first inequality in (6.15).

To estimate V_*, observe that

$$\begin{aligned}V_* &= \frac{1}{n-1}\sum_{j=1}^{n-1}|\zeta_j|^2 \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left|a-\frac{1}{q_j}\right|^2 \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1}|q_j|^{-2}-a^2,\end{aligned}$$

where we again used $(n-1)^{-1}\sum_j q_j^{-1}=a$. Furthermore,

$$\frac{1}{|q_j|^2}=1+\frac{1-|q_j|^2}{|q_j|^2}\le 1+\frac{1-|q_j|^2}{1-(n-1)(1-s)}$$

by (6.17). Averaging over j therefore gives

$$\frac{1}{n-1}\sum_{j=1}^{n-1}|q_j|^{-2}\le 1+\frac{1-s}{1-(n-1)(1-s)}.$$

Consequently,

$$V_*\le 1+\frac{1-s}{1-(n-1)(1-s)}-a^2,$$

which proves the second inequality in (6.15).

We now prove (6.16), for which no assumption on $(n-1)(1-s)$ is needed. Applying Lemma 6.5 with $N=n-1$ and $\xi_j=q_j$, we obtain

$$\left(\prod_{j=1}^{n-1}|q_j|\right)\sum_{j=1}^{n-1}\left|\frac{1}{q_j}-\overline{q_j}\right|\le 1-\prod_{j=1}^{n-1}|q_j|^2. \tag{6.18}$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{n-1}\left(\frac{1}{q_j}-\overline{q_j}\right) &= (n-1)a-(n-1)\overline{\mu} \\ &= (n-1)(a-\overline{\mu}),\end{aligned}$$

and hence, by the triangle inequality,

$$(n-1)|a-\mu| \le \sum_{j=1}^{n-1} \left| \frac{1}{q_j} - \overline{q_j} \right|. \tag{6.19}$$

Combining (6.18) and (6.19) yields

$$(n-1)|a-\mu| \prod_{j=1}^{n-1} |q_j| \le 1 - \prod_{j=1}^{n-1} |q_j|^2.$$

Since every $q_j \neq 0$, division by $\prod_{j=1}^{n-1} |q_j|$ gives

$$\begin{aligned}(n-1)|a-\mu| &\le \left(\prod_{j=1}^{n-1} |q_j| \right)^{-1} - \prod_{j=1}^{n-1} |q_j| \\ &= \left(\prod_{j=1}^{n-1} |q_j|^2 \right)^{-1/2} - \left(\prod_{j=1}^{n-1} |q_j|^2 \right)^{1/2},\end{aligned}$$

which is (6.16). □

Meir and Sharma [MS69, Lemma 2] used the same extremal principle to maximize an inverse-power sum under pointwise and product constraints. We shall need it here for two convex functions, so we include the short argument.

Lemma 6.7. *Suppose* $0 < L \le \prod_{j=1}^{n-1} |q_j|^2$, $s \le s_+$, *and let* $0 < \eta < s_+ \le 1$ *satisfy*

$$\eta \left(\frac{(n-1)s_+ - \eta}{n-2} \right)^{n-2} < L. \tag{6.20}$$

Then

$$|q_j|^2 > \eta, \qquad 1 \le j \le n-1.$$

Assume in addition that $L \le 1$, *and choose* $k \in \{0, \dots, n-2\}$ *such that* $\eta^{k+1} \le L \le \eta^k$. *Set* $z = L/\eta^k$. *Then*

$$(n-1)|a-\mu| \le M := \frac{k(1-\eta)}{\sqrt{\eta}} + \frac{1-z}{\sqrt{z}}, \tag{6.21}$$

$$V_* \le T - a^2, \qquad T := \frac{k/\eta + 1/z + n - k - 2}{n-1}. \tag{6.22}$$

Proof. We first prove the pointwise lower bound. Suppose, to the contrary, that $|q_j|^2 = x \le \eta$ for some j. Since

$$\sum_{\ell=1}^{n-1} |q_\ell|^2 = (n-1)s \le (n-1)s_+,$$

the arithmetic–geometric mean inequality applied to the remaining $n-2$ terms gives

$$\prod_{\ell=1}^{n-1} |q_\ell|^2 \le x \left(\frac{(n-1)s_+ - x}{n-2} \right)^{n-2}. \tag{6.23}$$

The function

$$x \longmapsto x \left(\frac{(n-1)s_+ - x}{n-2} \right)^{n-2}$$

is increasing on $0 \le x \le s_+$. Indeed, apart from a positive constant, its derivative has the sign of $(n-1)s_+ - (n-1)x$. Hence (6.23) and $x \le \eta$ imply

$$\prod_{\ell=1}^{n-1} |q_\ell|^2 \le \eta \left(\frac{(n-1)s_+ - \eta}{n-2} \right)^{n-2} < L,$$

contrary to the assumption $L \le \prod_{\ell=1}^{n-1} |q_\ell|^2$. Thus

$$|q_j|^2 > \eta \qquad (1 \le j \le n-1). \tag{6.24}$$

Notice also that $s_+ > \eta$ implies

$$\frac{(n-1)s_+ - \eta}{n-2} > \eta.$$

Therefore (6.20) gives $L > \eta^{n-1}$. Together with $L \le 1$, this guarantees the existence of $k \in \{0, \dots, n-2\}$ satisfying

$$\eta^{k+1} \le L \le \eta^k.$$

In particular,

$$\eta \le z = \frac{L}{\eta^k} \le 1.$$

We now turn to the extremal argument. Set $t_j = -\log |q_j|^2$, and $\lambda = -\log \eta$. By (6.24),

$$0 \le t_j < \lambda.$$

Moreover,

$$\sum_{j=1}^{n-1} t_j = -\log \prod_{j=1}^{n-1} |q_j|^2 \le -\log L.$$

Since $L > \eta^{n-1}$,

$$-\log L < (n-1)\lambda.$$

Let f be any convex nondecreasing function on $[0, \lambda]$. Because f is nondecreasing, enlarging $\sum_j t_j$ up to $-\log L$ cannot decrease $\sum_j f(t_j)$. We may therefore impose $\sum_{j=1}^{n-1} t_j = -\log L$. If two coordinates t_i, t_j both lie strictly between 0 and λ, move an amount $\delta = \min\{t_i, \lambda - t_j\}$ from the smaller coordinate to the larger one. Thus

$$(t_i, t_j) \longmapsto (t_i - \delta, t_j + \delta).$$

The sum is unchanged, while convexity of f shows that

$$f(t_i - \delta) + f(t_j + \delta) \ge f(t_i) + f(t_j).$$

Repeating this operation leaves at most one coordinate strictly between 0 and λ.

Since $\eta^{k+1} \le L \le \eta^k$, taking logarithms gives

$$k\lambda \le -\log L \le (k+1)\lambda.$$

Furthermore,

$$-\log L = k\lambda - \log z.$$

Thus the extremal vector has

$$k \text{ coordinates equal to } \lambda, \qquad 1 \text{ coordinate equal to } -\log z,$$

and the remaining $n-k-2$ coordinates equal to 0. Consequently, for every convex nondecreasing f,

$$\sum_{j=1}^{n-1} f(t_j) \le kf(\lambda) + f(-\log z) + (n-k-2)f(0). \tag{6.25}$$

We first apply (6.25) to $f(t) = 2\sinh(t/2)$. Since $1/q_j - \overline{q_j} = (1 - |q_j|^2)/q_j$, we have

$$\left| \frac{1}{q_j} - \overline{q_j} \right| = \frac{1 - |q_j|^2}{|q_j|} = 2\sinh(t_j/2).$$

Also,

$$\sum_{j=1}^{n-1} \left(\frac{1}{q_j} - \overline{q_j} \right) = (n-1)(a - \overline{\mu}).$$

Hence

$$(n-1)|a-\mu| \le \sum_{j=1}^{n-1} 2\sinh(t_j/2). \tag{6.26}$$

Now

$$2\sinh(\lambda/2) = \frac{1-\eta}{\sqrt{\eta}}, \qquad 2\sinh\left(\frac{-\log z}{2}\right) = \frac{1-z}{\sqrt{z}}.$$

Applying (6.25) to (6.26) therefore gives

$$(n-1)|a-\mu| \le \frac{k(1-\eta)}{\sqrt{\eta}} + \frac{1-z}{\sqrt{z}},$$

which proves (6.21).

For the variance estimate, apply (6.25) to $f(t) = e^t$. Since $e^{t_j} = |q_j|^{-2}$, and $\frac{1}{n-1}\sum_{j=1}^{n-1}|q_j|^{-2} = a^2 + V_*$, we obtain

$$\begin{aligned}(n-1)(a^2+V_*) &= \sum_{j=1}^{n-1} e^{t_j} \\ &\le \frac{k}{\eta} + \frac{1}{z} + n - k - 2.\end{aligned}$$

Dividing by $n-1$ and subtracting a^2 gives

$$V_* \le \frac{k/\eta + 1/z + n - k - 2}{n-1} - a^2 = T - a^2,$$

which is (6.22). □

7. Centered coefficients and configurations near equality

The centered interpolation estimates control the full product in terms of the variance v. For the small degrees, however, it is useful to retain the individual coefficients of the centered product. Write

$$\boldsymbol{w} = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}.$$

Repeated application of Schoenberg's inequality gives the coefficient estimate needed below.

Lemma 7.1. *For $2 \le k \le n-1$,*

$$|e_k(\boldsymbol{w})| \le \binom{n-1}{k}\left(\frac{k-1}{n-2}v\right)^{k/2}. \tag{7.1}$$

Proof. Let $R(z) = \prod_{j=1}^{n-1}(z-w_j)$. Since $\sum_{j=1}^{n-1} w_j = 0$, the centroid of the zeros of R is zero. The same is true for every derivative of R.

Fix $2 \le k \le n-2$, and let $\lambda_1, \dots, \lambda_k$ be the zeros of $R^{(n-1-k)}$. Applying Schoenberg's inequality (2.2) successively to $R, R', \dots, R^{(n-k-2)}$, whose degrees are $n-1, n-2, \dots, k+1$, gives

$$\begin{aligned}\sum_{j=1}^{k}|\lambda_j|^2 &\le \left(\prod_{m=k+1}^{n-1}\frac{m-2}{m}\right)\sum_{j=1}^{n-1}|w_j|^2 \\ &= \frac{k(k-1)}{(n-1)(n-2)}(n-1)v.\end{aligned}$$

Thus

$$\sum_{j=1}^{k}|\lambda_j|^2 \le \frac{k(k-1)}{n-2}v. \tag{7.2}$$

Next, write

$$R(z) = z^{n-1} + \sum_{r=1}^{n-1}(-1)^r e_r(\boldsymbol{w}) z^{n-1-r}.$$

The leading coefficient of $R^{(n-1-k)}$ is $(n-1)!/k!$, whereas its constant term is $(-1)^k(n-1-k)!e_k(\boldsymbol{w})$. Hence the constant term after making $R^{(n-1-k)}$ monic is

$$(-1)^k \frac{e_k(\boldsymbol{w})}{\binom{n-1}{k}}. \tag{7.3}$$

Since the product of the zeros of this monic polynomial is $(-1)^k$ times its constant term, (7.3) gives

$$\frac{|e_k(\boldsymbol{w})|}{\binom{n-1}{k}} = \prod_{j=1}^{k} |\lambda_j|.$$

The arithmetic–geometric mean inequality, followed by (7.2), now yields

$$\begin{aligned}\frac{|e_k(\boldsymbol{w})|}{\binom{n-1}{k}} &\le \left(\frac{1}{k}\sum_{j=1}^{k}|\lambda_j|^2\right)^{k/2} \\ &\le \left(\frac{k-1}{n-2}v\right)^{k/2}.\end{aligned}$$

This proves (7.1) for $2 \le k \le n-2$.

For $k = n-1$, no differentiation is required. Since $e_{n-1}(\boldsymbol{w}) = \prod_{j=1}^{n-1} w_j$, the arithmetic–geometric mean inequality gives

$$|e_{n-1}(\boldsymbol{w})| \le \left(\frac{1}{n-1}\sum_{j=1}^{n-1}|w_j|^2\right)^{(n-1)/2} = v^{(n-1)/2},$$

which is precisely (7.1) in this case. □

For nonnegative integers k, p and $d \ge 0$, define

$$S_{k,p}(d) = \sum_{j=0}^{p}\binom{k+j}{k}d^j, \qquad I_{k,p}(d) = \frac{S_{k,p}(d)}{(k+p+1)\binom{k+p}{k}} = \int_0^1 t^k(1-t+dt)^p\,\mathrm{d}t. \tag{7.4}$$

To verify the integral identity, expand with respect to d:

$$(1-t+dt)^p = \sum_{j=0}^{p}\binom{p}{j}d^j t^j(1-t)^{p-j}.$$

Using the beta integral,

$$\begin{aligned}\int_0^1 t^k(1-t+dt)^p\,\mathrm{d}t &= \sum_{j=0}^{p}\binom{p}{j}d^j\int_0^1 t^{k+j}(1-t)^{p-j}\,\mathrm{d}t \\ &= \frac{1}{(k+p+1)\binom{k+p}{k}}\sum_{j=0}^{p}\binom{k+j}{k}d^j,\end{aligned}$$

which is the second identity in (7.4).

Lemma 7.2. *If $\mathcal{C} \ge |a\mu J|$ and $d \ge |1-a\mu|$, then every normalized counterexample satisfies*

$$1 \le \mathcal{C} + d^n + a|\mu|\sum_{k=2}^{n-1}\left(\frac{a^2 v(k-1)}{n-2}\right)^{k/2} S_{k,n-1-k}(d). \tag{7.5}$$

Proof. Since $q_j = \mu + w_j$ and $e_1(\boldsymbol{w}) = 0$, expansion of the product gives

$$F(t) = (1-a\mu t)^{n-1} + \sum_{k=2}^{n-1}(-at)^k e_k(\boldsymbol{w})(1-a\mu t)^{n-1-k}. \tag{7.6}$$

We first establish a convenient bound for the remaining linear factor. Since

$$1 - a\mu t = (1-t) + t(1-a\mu),$$

the triangle inequality and the assumption $d \geq |1 - a\mu|$ give

$$|1 - a\mu t| \leq 1 - t + dt, \qquad 0 \leq t \leq 1. \tag{7.7}$$

Taking absolute values in (3.6) and using $|a\mu J| \leq \mathcal{C}$ and $|1 - a\mu| \leq d$, we obtain

$$1 \leq \mathcal{C} + d^n + na|\mu| \int_0^1 |F(t) - (1 - a\mu t)^{n-1}| \, \mathrm{d}t. \tag{7.8}$$

By (7.6) and (7.7),

$$|F(t) - (1 - a\mu t)^{n-1}| \leq \sum_{k=2}^{n-1} a^k t^k |e_k(\boldsymbol{w})| (1 - t + dt)^{n-1-k}.$$

Using Lemma 7.1 and integrating term by term therefore gives

$$\begin{aligned} &na|\mu| \int_0^1 |F(t) - (1 - a\mu t)^{n-1}| \, \mathrm{d}t \\ &\quad \leq na|\mu| \sum_{k=2}^{n-1} a^k \binom{n-1}{k} \left(\frac{k-1}{n-2} v \right)^{k/2} I_{k,n-1-k}(d). \end{aligned}$$

Since $k + (n - 1 - k) + 1 = n$, (7.4) gives

$$n \binom{n-1}{k} I_{k,n-1-k}(d) = S_{k,n-1-k}(d). \tag{7.9}$$

Consequently,

$$\begin{aligned} &na|\mu| \int_0^1 |F(t) - (1 - a\mu t)^{n-1}| \, \mathrm{d}t \\ &\quad \leq a|\mu| \sum_{k=2}^{n-1} \left(\frac{a^2 v(k-1)}{n-2} \right)^{k/2} S_{k,n-1-k}(d). \end{aligned}$$

Substituting this estimate into (7.8) proves (7.5). □

7.1. **Retaining the quadratic term.** The coefficient estimate in Lemma 7.2 treats all centered coefficients by absolute value. Near the limiting configuration, however, the quadratic term carries a favorable sign. We now retain this information.

Assume that $\mu \neq 0$, and write $\mu = |\mu| e^{i\theta}$. Set

$$q'_j = e^{-i\theta} q_j, \qquad w'_j = q'_j - |\mu|, \qquad \boldsymbol{w}' = (w'_1, \ldots, w'_{n-1}).$$

Since $\frac{1}{n-1} \sum_{j=1}^{n-1} q_j = \mu$, we have

$$\frac{1}{n-1} \sum_{j=1}^{n-1} q'_j = |\mu|, \qquad \sum_{j=1}^{n-1} w'_j = 0.$$

Moreover, $w'_j = e^{-i\theta} w_j$, so that

$$\frac{1}{n-1} \sum_{j=1}^{n-1} |w'_j|^2 = v.$$

Define

$$W = \frac{1}{n-1} \sum_{j=1}^{n-1} (\operatorname{Re} q'_j - |\mu|)^2 = \frac{1}{n-1} \sum_{j=1}^{n-1} (\operatorname{Re} w'_j)^2.$$

Since $e_1(\boldsymbol{w}') = 0$,

$$e_2(\boldsymbol{w}') = -\frac{1}{2} \sum_{j=1}^{n-1} (w'_j)^2.$$

Writing $w_j' = \operatorname{Re} w_j' + i \operatorname{Im} w_j'$, we therefore obtain

$$\begin{aligned}\operatorname{Re} e_2(\boldsymbol{w}') &= -\frac{1}{2}\sum_{j=1}^{n-1}\Big((\operatorname{Re} w_j')^2 - (\operatorname{Im} w_j')^2\Big)\\ &= \frac{1}{2}\sum_{j=1}^{n-1}|w_j'|^2 - \sum_{j=1}^{n-1}(\operatorname{Re} w_j')^2.\end{aligned}$$

Hence

$$\operatorname{Re} e_2(\boldsymbol{w}') = \frac{n-1}{2}v - (n-1)W. \tag{7.10}$$

To estimate W, we shall use the following classical variance bound.

Lemma 7.3 ([BD00, p. 353])**.** *Let $\xi_1, \dots, \xi_N \in [L, U]$, and let $\omega_1, \dots, \omega_N \geq 0$ satisfy $\sum_{j=1}^N \omega_j = 1$. If $b = \sum_{j=1}^N \omega_j \xi_j$, then*

$$\sum_{j=1}^N \omega_j(\xi_j - b)^2 \leq (U-b)(b-L).$$

For the reciprocal critical-point coordinates, the relevant interval can be bounded explicitly.

Lemma 7.4. *Suppose that $\mu \neq 0$ and $|q_j|^2 \geq t_0 \geq 0, 1 \leq j \leq n-1$. Set*

$$L_\mu = \max\left\{-1,\ (n-1)|\mu| - (n-2),\ \frac{(n-1)^2|\mu|^2 + t_0 - (n-2)^2}{2(n-1)|\mu|}\right\}. \tag{7.11}$$

Then

$$\operatorname{Re} q_j' \geq L_\mu, \qquad W \leq \min\Big\{(n-2)(1-|\mu|)^2,\ (1-|\mu|)(|\mu| - L_\mu)\Big\}. \tag{7.12}$$

Proof. We first establish the lower bound for $\operatorname{Re} q_j'$. Since $|q_j'| = |q_j| < 1$, $\operatorname{Re} q_j' \geq -1$. Also, $\sum_{j=1}^{n-1} q_j' = (n-1)|\mu|$. Since $\operatorname{Re} q_k' \leq 1$ for every k,

$$\operatorname{Re} q_j' = (n-1)|\mu| - \sum_{k\neq j}\operatorname{Re} q_k' \geq (n-1)|\mu| - (n-2). \tag{7.13}$$

Finally, $(n-1)|\mu| - q_j' = \sum_{k\neq j} q_k'$, and hence

$$\Big|(n-1)|\mu| - q_j'\Big| \leq \sum_{k\neq j}|q_k'| \leq n-2.$$

Squaring gives

$$(n-1)^2|\mu|^2 + |q_j'|^2 - 2(n-1)|\mu|\operatorname{Re} q_j' \leq (n-2)^2.$$

Since $|q_j'|^2 = |q_j|^2 \geq t_0$, it follows that

$$\operatorname{Re} q_j' \geq \frac{(n-1)^2|\mu|^2 + t_0 - (n-2)^2}{2(n-1)|\mu|}. \tag{7.14}$$

Combining $\operatorname{Re} q_j' \geq -1$, (7.13), and (7.14) proves

$$\operatorname{Re} q_j' \geq L_\mu.$$

We next bound W. Apply Lemma 7.3 with

$$N = n-1, \qquad Y_j = \operatorname{Re} q_j', \qquad \omega_j = \frac{1}{n-1},$$

and

$$L = L_\mu, \qquad U = 1, \qquad b = |\mu|.$$

Since $\frac{1}{n-1}\sum_{j=1}^{n-1}\operatorname{Re} q_j' = |\mu|$, we obtain

$$W \leq (1-|\mu|)(|\mu| - L_\mu). \tag{7.15}$$

For the second estimate, put $d_j = 1 - \operatorname{Re} q_j' \geq 0$. Then

$$\sum_{j=1}^{n-1} d_j = (n-1)(1-|\mu|),$$

and

$$\begin{aligned} W &= \frac{1}{n-1}\sum_{j=1}^{n-1}\left(d_j - (1-|\mu|)\right)^2 \\ &= \frac{1}{n-1}\sum_{j=1}^{n-1} d_j^2 - (1-|\mu|)^2. \end{aligned}$$

Since $d_j \geq 0$,

$$\sum_{j=1}^{n-1} d_j^2 \leq \left(\sum_{j=1}^{n-1} d_j\right)^2,$$

and therefore

$$W \leq (n-2)(1-|\mu|)^2. \tag{7.16}$$

Combining (7.15) and (7.16) proves (7.12). □

Define

$$K_2(z) = nz\int_0^1 t^2(1-zt)^{n-3}\,\mathrm{d}t. \tag{7.17}$$

The quadratic term in (7.6) contributes $-a^2 e_2(\boldsymbol{w})K_2(a\mu)$ to the last term of (3.6). Instead of estimating this contribution by its absolute value, we shall retain its real part.

Lemma 7.5. *Suppose* $0 < h_0 := a|\mu| \leq 1, \varepsilon \geq |e^{i\theta}-1|$, *and* $d \geq |1-a\mu|$. *Then*

$$|e^{2i\theta}K_2(a\mu) - K_2(h_0)| \leq E_2, \tag{7.18}$$

where

$$E_2 := nh_0\varepsilon\left\{3I_{2,n-3}(d) + (n-3)h_0 I_{3,n-4}(d)\right\}.$$

If, in addition, $0 < L_2 \leq \operatorname{Re} e_2(\boldsymbol{w}'), 0 \leq K_- \leq K_2(h_0)$, *and* $\mathcal{C} \geq |a\mu J|$, *then*

$$\begin{aligned} 1 \leq \mathcal{C} + d^n - a^2K_-L_2 + \frac{(n-1)a^2v}{2}E_2 \\ + a|\mu|\sum_{k=3}^{n-1}\left(\frac{a^2v(k-1)}{n-2}\right)^{k/2} S_{k,n-1-k}(d). \end{aligned} \tag{7.19}$$

Proof. Since $a\mu = h_0e^{i\theta}$, the definition (7.17) gives

$$e^{2i\theta}K_2(a\mu) = nh_0e^{3i\theta}\int_0^1 t^2(1-h_0e^{i\theta}t)^{n-3}\,\mathrm{d}t.$$

Moreover,

$$K_2(h_0) = nh_0\int_0^1 t^2(1-h_0t)^{n-3}\,\mathrm{d}t.$$

Since $0 < h_0 \leq 1$,

$$0 \leq 1 - h_0t \leq |1-h_0e^{i\theta}t| = |1-a\mu t|.$$

Also, $1 - a\mu t = (1-t) + t(1-a\mu)$, so the assumption $d \geq |1-a\mu|$ gives

$$|1-a\mu t| \leq 1 - t + dt. \tag{7.20}$$

Set $A = 1 - h_0e^{i\theta}t$, and $B = 1 - h_0t$. Then

$$|e^{3i\theta}A^{n-3} - B^{n-3}| \leq |e^{3i\theta}-1|\,|A|^{n-3} + |A^{n-3} - B^{n-3}|. \tag{7.21}$$

For the first term on the right-hand side of (7.21), we have

$$|e^{3i\theta}-1| \leq 3|e^{i\theta}-1| \leq 3\varepsilon.$$

Moreover, by (7.20),

$$|A| = |1 - a\mu t| \le 1 - t + dt.$$

Hence

$$|e^{3i\theta} - 1|\,|A|^{n-3} \le 3\varepsilon(1 - t + dt)^{n-3}. \tag{7.22}$$

For the second term on the right-hand side of (7.21), telescoping the difference of powers gives

$$\begin{aligned} |A^{n-3} - B^{n-3}| &\le (n-3)|A - B| \max\{|A|, |B|\}^{n-4} \\ &\le (n-3)h_0 t\varepsilon(1 - t + dt)^{n-4}. \end{aligned} \tag{7.23}$$

Substituting (7.22) and (7.23) into (7.21) yields

$$\begin{aligned} |e^{3i\theta}A^{n-3} - B^{n-3}| &\le 3\varepsilon(1 - t + dt)^{n-3} \\ &\quad + (n-3)h_0 t\varepsilon(1 - t + dt)^{n-4}. \end{aligned}$$

Therefore

$$\begin{aligned} &|e^{2i\theta}K_2(a\mu) - K_2(h_0)| \\ &\quad \le nh_0\varepsilon\left\{3\int_0^1 t^2(1 - t + dt)^{n-3}\,\mathrm{d}t + (n-3)h_0\int_0^1 t^3(1 - t + dt)^{n-4}\,\mathrm{d}t\right\}. \end{aligned}$$

By (7.4), the right-hand side is precisely E_2, proving (7.18).

We now retain the sign of the quadratic contribution. Since $w_j = e^{i\theta}w_j'$, we have

$$e_2(\boldsymbol{w}) = e^{2i\theta}e_2(\boldsymbol{w}').$$

Consequently,

$$\begin{aligned} \mathrm{Re}(e_2(\boldsymbol{w})K_2(a\mu)) &= \mathrm{Re}\Big(e_2(\boldsymbol{w}')e^{2i\theta}K_2(a\mu)\Big) \\ &\ge K_2(h_0)\,\mathrm{Re}\,e_2(\boldsymbol{w}') - |e_2(\boldsymbol{w}')|E_2. \end{aligned}$$

Using $K_2(h_0) \ge K_-, \mathrm{Re}\,e_2(\boldsymbol{w}') \ge L_2$, and Lemma 7.1 with $k = 2$,

$$|e_2(\boldsymbol{w}')| \le \frac{n-1}{2}v,$$

we obtain

$$\mathrm{Re}(e_2(\boldsymbol{w})K_2(a\mu)) \ge K_-L_2 - \frac{n-1}{2}vE_2. \tag{7.24}$$

Finally, take real parts in (3.6). Separating the quadratic term in (7.6) gives

$$\begin{aligned} 1 &\le |a\mu J| + |1 - a\mu|^n - a^2\,\mathrm{Re}(e_2(\boldsymbol{w})K_2(a\mu)) \\ &\quad + na|\mu|\sum_{k=3}^{n-1} a^k|e_k(\boldsymbol{w})|I_{k,n-1-k}(d). \end{aligned}$$

Using $|a\mu J| \le \mathcal{C}, |1 - a\mu| \le d$, the lower bound (7.24), and Lemma 7.1, we obtain

$$\begin{aligned} 1 &\le \mathcal{C} + d^n - a^2K_-L_2 + \frac{(n-1)a^2v}{2}E_2 \\ &\quad + na|\mu|\sum_{k=3}^{n-1} a^k\binom{n-1}{k}\left(\frac{k-1}{n-2}v\right)^{k/2} I_{k,n-1-k}(d). \end{aligned}$$

Finally, (7.9) transforms the last sum into

$$a|\mu|\sum_{k=3}^{n-1}\left(\frac{a^2v(k-1)}{n-2}\right)^{k/2} S_{k,n-1-k}(d),$$

which proves (7.19). □

7.2. **An analytic neighborhood of $|\mu| = 1$.** The finite verification must not leave an unchecked neighborhood of the limiting equality configuration $|\mu| = 1$. The next lemma excludes such a neighborhood uniformly for the small degrees, without any computer-assisted verification.

Lemma 7.6. *There is no normalized counterexample with $4 \le n \le 13$ and $|\mu| \ge \frac{99999}{100000}$.*

Proof. Put $\delta = 1 - |\mu|$. Then $0 < \delta \le 10^{-5}$. Since $s \ge |\mu|^2$,

$$1 - s \le 1 - |\mu|^2 = 2\delta - \delta^2 \le 2\delta.$$

Step 1. Control of a and the phase. Since $\sum_{j=1}^{n-1}(1 - |q_j|^2) = (n-1)(1-s)$, each summand satisfies $1 - |q_j|^2 \le (n-1)(1-s)$. As $n \le 13$, we obtain

$$|q_j|^2 \ge 1 - (n-1)(1-s) \ge 1 - 24\delta > \left(\frac{99}{100}\right)^2.$$

Moreover,

$$(n-1)(1-s) \le 24\delta < 1,$$

so the first estimate in Lemma 6.6 applies. Using $1 - s \le 2\delta$, we obtain

$$|a - \mu| \le \frac{1-s}{\sqrt{1-(n-1)(1-s)}} \le \frac{2\delta}{\sqrt{1-24\delta}} < \frac{21}{10}\delta. \tag{7.25}$$

Since $\big|a - |\mu|\big| \le |a - \mu|$, and $|\mu| = 1 - \delta$, (7.25) gives

$$1 - 4\delta \le a \le 1 + 2\delta.$$

In particular, $a^2 < 2$.

We next control the phase of μ. Since $\big|\mu - |\mu|\big| \le |\mu - a| + \big|a - |\mu|\big|$, equation (7.25) yields

$$\big|\mu - |\mu|\big| < \frac{21}{5}\delta.$$

Writing $\mu = |\mu|e^{i\theta}$, we therefore have

$$|e^{i\theta} - 1| = \frac{\big|\mu - |\mu|\big|}{|\mu|} < \frac{(21/5)\delta}{1-\delta} < 5\delta.$$

Finally,

$$\big|a|\mu| - 1\big| \le |\mu|\big|a - |\mu|\big| + (1 - |\mu|^2) < \frac{21}{10}\delta + 2\delta,$$

and hence

$$|a\mu - 1| \le a\big|\mu - |\mu|\big| + \big|a|\mu| - 1\big| < 12\delta.$$

We have thus proved

$$1 - 4\delta \le a \le 1 + 2\delta, \qquad a^2 < 2, \qquad |e^{i\theta} - 1| < 5\delta, \qquad |a\mu - 1| < 12\delta. \tag{7.26}$$

For later use, observe also that $1 - a\mu t = (1-t) + t(1 - a\mu)$. Hence, by (7.26),

$$|1 - a\mu t| \le 1 - t + 12\delta t \le 1, \qquad 0 \le t \le 1. \tag{7.27}$$

Step 2. The quadratic contribution. From (7.17),

$$K_2(1) = n\int_0^1 t^2(1-t)^{n-3}\,\mathrm{d}t = \frac{2}{(n-1)(n-2)}.$$

We first compare $K_2(a\mu)$ with $K_2(1)$. By (7.27) and telescoping the difference of powers,

$$\begin{aligned}|K_2(a\mu) - K_2(1)| &\le n|a\mu - 1|\left(\int_0^1 t^2\,\mathrm{d}t + (n-3)\int_0^1 t^3\,\mathrm{d}t\right)\\ &= n|a\mu - 1|\left(\frac{1}{3} + \frac{n-3}{4}\right).\end{aligned}$$

Using $n \le 13$ and (7.26), we obtain

$$|K_2(a\mu) - K_2(1)| \le 442\delta. \tag{7.28}$$

Moreover,

$$|e^{2i\theta}-1| \le 2|e^{i\theta}-1| < 10\delta,$$

while

$$K_2(1) = \frac{2}{(n-1)(n-2)} \le \frac{1}{3}.$$

Therefore (7.28) gives

$$|e^{2i\theta}K_2(a\mu) - K_2(1)| < 450\delta. \tag{7.29}$$

Recall that $e_2(\boldsymbol{w}) = e^{2i\theta}e_2(\boldsymbol{w}')$. Hence

$$\operatorname{Re}(e_2(\boldsymbol{w})K_2(a\mu)) \ge K_2(1)\operatorname{Re} e_2(\boldsymbol{w}') - 450\delta\,|e_2(\boldsymbol{w}')|.$$

By (7.10),

$$\operatorname{Re} e_2(\boldsymbol{w}') = \frac{n-1}{2}v - (n-1)W,$$

and the first bound in (7.12) gives

$$W \le (n-2)\delta^2.$$

Consequently,

$$\begin{aligned} K_2(1)\operatorname{Re} e_2(\boldsymbol{w}') &= \frac{v}{n-2} - \frac{2W}{n-2} \\ &\ge \frac{v}{n-2} - 2\delta^2. \end{aligned}$$

Also, Lemma 7.1 with $k=2$ gives

$$|e_2(\boldsymbol{w}')| \le \frac{n-1}{2}v.$$

Combining these estimates with (7.29), we obtain

$$\operatorname{Re}(e_2(\boldsymbol{w})K_2(a\mu)) \ge \frac{v}{n-2} - 2\delta^2 - 225(n-1)\delta v \ge -2\delta^2. \tag{7.30}$$

For the last inequality, it is enough to note that, for $4 \le n \le 13$ and $\delta \le 10^{-5}$,

$$\frac{1}{n-2} \ge \frac{1}{11} > 225\cdot 12\cdot 10^{-5} \ge 225(n-1)\delta.$$

Step 3. The higher-order coefficients. Set $\tau = a\sqrt{v}$. Since $v \le 1-|\mu|^2 \le 2\delta$ and $a^2 < 2$, we have

$$\tau \le 2\sqrt{2\delta} < \frac{1}{100}. \tag{7.31}$$

For the terms of order $k \ge 3$, we use the coefficient bounds from the proof of Lemma 7.2, with $d = 12\delta$. For $3 \le k \le n-1$, the binomial series gives

$$\begin{aligned} S_{k,n-1-k}(12\delta) &\le \sum_{j=0}^{\infty}\binom{k+j}{k}(12\delta)^j \\ &= (1-12\delta)^{-k-1} \\ &< 2, \end{aligned}$$

because $k \le n-1 \le 12$ and $\delta \le 10^{-5}$.

Furthermore, (7.26) gives $|a\mu| < 2$, and for $3 \le k \le n-1$,

$$\left(\frac{k-1}{n-2}\right)^{k/2} \le 1.$$

Hence the total contribution of the terms with $k \ge 3$ is bounded by

$$\begin{aligned} a|\mu|\sum_{k=3}^{n-1}\left(\frac{a^2v(k-1)}{n-2}\right)^{k/2} S_{k,n-1-k}(12\delta) &\le 4\sum_{k=3}^{n-1}\tau^k < 4\sum_{k=3}^{\infty}\tau^k \\ &< 5\tau^3 < 128\delta^{3/2}, \end{aligned}$$

where the last two inequalities follow from (7.31). Thus

$$a|\mu| \sum_{k=3}^{n-1} \left(\frac{a^2 v(k-1)}{n-2} \right)^{k/2} S_{k,n-1-k}(12\delta) < 128\delta^{3/2}. \tag{7.32}$$

We now return to the centered identity. By (3.7), $|a\mu J| \le |\mu| = 1-\delta$. Also, by (7.26),

$$|1-a\mu|^n \le (12\delta)^n \le (12\delta)^4.$$

The quadratic contribution satisfies, by (7.30) and $a^2 < 2$,

$$-a^2 \operatorname{Re}(e_2(\boldsymbol{w}) K_2(a\mu)) \le 4\delta^2.$$

Combining these estimates with (7.32) in (3.6), we obtain

$$1 \le 1 - \delta + 4\delta^2 + 128\delta^{3/2} + (12\delta)^4. \tag{7.33}$$

But (7.33) would require

$$1 \le 4\delta + 128\sqrt{\delta} + 20736\delta^3.$$

This is impossible, since

$$4\delta + 128\sqrt{\delta} + 20736\delta^3 < \frac{4}{10^5} + \frac{128}{300} + \frac{20736}{10^{15}} < \frac{1}{2}.$$

The contradiction proves the lemma. □

8. Finite verification and completion of the proof

Before carrying out the finite verification, we reduce the location of the distinguished zero to a fixed bounded interval. This reduction is independent of the large-degree cutoff in Proposition 5.1.

Lemma 8.1. *Every normalized counterexample satisfies* $0 < a < 5$.

Proof. We already know that $a > 0$. Suppose, for contradiction, that $a \ge 5$. Taking $V = 1$ in the second inequality of (3.1) gives

$$|\mu - a|^2 \le a^2(1-s).$$

Expanding the left-hand side, we obtain

$$a^2 s \le 2ax - |\mu|^2 \le 2a|\mu|.$$

On the other hand, the first inequality in (3.2) gives

$$|1-a\mu|^2 \le v \le s,$$

and hence

$$a|\mu| \le 1 + |1-a\mu| \le 1 + \sqrt{s}.$$

Writing $s_0 = \sqrt{s}$, we therefore have

$$a^2 s_0^2 \le 2 + 2s_0.$$

Since $a \ge 5$,

$$25 s_0^2 \le 2 + 2s_0,$$

which implies $s_0 < 1/3$. Consequently,

$$s < \frac{1}{9}, \qquad a^2 s \le 2a|\mu| \le 2 + 2\sqrt{s} < \frac{8}{3}. \tag{8.1}$$

By (4.4), with $V = 1$,

$$\frac{1}{n-1} \sum_{j=1}^{n-1} |1 - aq_j| \le \sqrt{s} < \frac{1}{3}.$$

We may therefore take

$$\kappa = \frac{1}{3}, \qquad \ell(t) = 1 - \frac{2}{3}t.$$

The arithmetic–geometric mean inequality then gives

$$|F(1)| = \prod_{j=1}^{n-1} |1 - aq_j| \le 3^{-(n-1)}. \tag{8.2}$$

For the remainder term, use the linear branch of (4.6). Since $\widetilde{c}_n < 3$, (8.1) gives

$$\begin{aligned} |\mathcal{R}| &\le a^2(n-1)\widetilde{c}_n s \int_0^1 t\left(1 - \frac{2}{3}t\right)^{n-2} \mathrm{d}t \\ &< 8(n-1)\int_0^{3/2} t\left(1 - \frac{2}{3}t\right)^{n-2} \mathrm{d}t \\ &= \frac{18}{n}. \end{aligned}$$

Thus

$$|\mathcal{R}| \le \frac{18}{n}. \tag{8.3}$$

Finally, (3.7), together with $\Phi_{n-1}(a) \le 1$, gives $|a\mu J| \le |\mu| \prod_{j=1}^{n-1} |q_j|$. Since $|\mu| \le \sqrt{s}$ and $\prod_{j=1}^{n-1} |q_j| \le s^{(n-1)/2}$, we obtain

$$|a\mu J| \le s^{n/2} < 3^{-n}. \tag{8.4}$$

The normalization (2.1) gives $a^2 \le n-1$. Thus $a \ge 5$ forces $n \ge 26$. Taking absolute values in (3.5) and using (3.4), (8.2), (8.3), and (8.4), we find

$$\begin{aligned} 1 &\le |F(1)| + \frac{n-1}{n}|a\mu J| + |\mathcal{R}| \\ &\le 3^{-(n-1)} + 3^{-n} + \frac{18}{n} \\ &\le 3^{-25} + 3^{-26} + \frac{9}{13} < 1, \end{aligned}$$

a contradiction. Hence $a < 5$. □

It follows that every normalized counterexample lies in the closed parameter box

$$(a, |\mu|, s) \in [0,5] \times [0,1] \times [0,1]. \tag{8.5}$$

The inclusion of the endpoints is deliberate. Although admissible data satisfy stricter inequalities, the closed box is more convenient for the finite subdivision, and ensures that common boundaries of adjacent boxes are covered automatically.

Proposition 8.2. *There is no normalized counterexample with* $4 \le n \le 100000$.

Proof. The finite verification is encoded by the two certificate files `certificate_n4_to_n13.json` and `certificate_n14_to_n100000.json`. The first treats the fixed degrees $4 \le n \le 13$, while the second treats $14 \le n \le 100000$ in consecutive degree blocks.

Each certificate describes a finite binary partition of (8.5). Every internal node bisects one of the three coordinate intervals at its exact midpoint, and every leaf records a single exclusion test from Appendix A. The corresponding exact-arithmetic checkers are `verify_low.py`, `verify_mid.py`, and `verify_all.py`.

The checkers reconstruct every box from its path using exact rational arithmetic. They reject malformed trees, verify that each internal node has both children, and hence certify that the leaves cover the entire parameter box. They also verify that the degree blocks are consecutive, with no omissions or overlaps, from degree 4 through degree 100000.

For every leaf, one of the following occurs. Either a necessary condition from Lemmas 3.1, 3.3, and 6.3–6.7 fails strictly; or Lemma 7.6 applies; or one of the necessary estimates (4.16), (4.17), (4.18), (7.5), and (7.19) has a right-hand side strictly smaller than 1. The interval bounds used in these tests, including the direction of every degree-dependent estimate, are specified in Appendix A. Appendix B shows that all numerical evaluations are performed with

directed exact-integer arithmetic, so every accepted leaf excludes the entire box represented by that leaf.

At precision $S = 2^{100}$, the complete verification gives the following counts.

Degrees n	Degree blocks	Leaf boxes	Maximum depth
4–13	10	12821	53
14–100000	62	20850	33
Total	72	33671	

The reported margins below concern the tests whose final rigorous upper bound is compared with 1; the elementary rational contradictions and applications of Lemma 7.6 are verified separately.

If such an upper bound is represented as U/S, the smallest positive values of $S - U$ in the low-degree and block verifications are, respectively,

$$859210771742754798767913 \quad \text{and} \quad 3409997860291474113416360. \tag{8.6}$$

Thus every accepted numerical test has strictly positive separation from its threshold.

The same certificates also pass at precision 2^{160}. The exact commands, file hashes, and verification details are recorded in Appendix B and in the supplementary repository at commit `83ee46a7c971`. Since the verified partitions cover (8.5), Lemma 8.1 completes the exclusion of all normalized counterexamples with $4 \le n \le 100000$. □

We can now complete the proof of the main theorem.

Proof of Theorem 1.2. If $\sigma_2(P) = 0$, then all zeros of P coincide, and (1.1) is immediate. If the prescribed zero is multiple, then it is itself a critical point, so there is again nothing to prove.

The cases $n = 2, 3$ follow from [KPPSS11, Theorem 3.2]. Suppose therefore that $n \ge 4$, that $\sigma_2(P) > 0$, and that the prescribed zero is simple. If (1.1) failed, the polynomial could be normalized as in (2.1). In the normalized setting, $a = 0$ is impossible: otherwise $|\zeta_j| > 1$ for every critical point, contradicting

$$V_* = \frac{1}{n-1} \sum_{j=1}^{n-1} |\zeta_j|^2 \le \frac{n-2}{n-1} < 1.$$

Thus $a > 0$.

Proposition 8.2 excludes $4 \le n \le 100000$, while Proposition 5.1 excludes $n \ge 100001$. Hence no counterexample exists, and (1.1) follows for every degree $n \ge 2$.

The constant in (1.1) is sharp. Indeed, for $P(z) = z^n - 1$, the centroid of the zeros is 0, and $\sigma_2(P)^2 = \frac{1}{n}\sum_{j=1}^{n} |z_j|^2 = 1$. The only critical point is 0, while every zero of P has modulus 1. Thus every zero is at distance exactly $\sigma_2(P) = 1$ from the unique critical point. □

Data availability

The certificates, exact checkers, optional generators, and verification logs are available in the supplementary repository [ZhaS]. This paper uses the files at commit `83ee46a7c971`; the certificate digests are recorded in Appendix B. These certificates are an essential part of Proposition 8.2. Reproduction requires only the Python standard library.

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Appendix A. Bounds on a parameter box

This appendix gives the mathematical specification of the finite verification used in Proposition 8.2. The labels attached to the tests in the certificate files are program identifiers

only; they do not introduce additional mathematical assertions. Throughout the appendix, an *admissible point* means a triple of parameters arising from a normalized counterexample.

For convenience, write $r := |\mu|$. A parameter box will be denoted by

$$\mathcal{B} = [a_-, a_+] \times [r_-, r_+] \times [s_-, s_+],$$

so that every admissible point in $\mathcal{B}$ satisfies

$$a_- \le a \le a_+, \qquad r_- \le r = |\mu| \le r_+, \qquad s_- \le s \le s_+.$$

Define

$$v_+ = s_+ - r_-^2, \qquad v_- = (s_- - r_+^2)_+,$$
$$d_a = \max\{0, a_- - r_+, r_- - a_+\},$$
$$d_h = \max\{0, 1 - a_+ r_+, a_- r_- - 1\}, \qquad \Delta_+ = 1 - s_-.$$

Since $v = s - r^2$, we have

$$v_- \le v \le v_+.$$

Moreover, $|a - \mu| \ge |a - r| \ge d_a$ and $|1 - a\mu| \ge |1 - ar| \ge d_h$. Finally, $1 - s \le \Delta_+$.

All interval estimates below are interpreted under the assumption that the box under consideration contains an admissible point.

A.1. **Refinements for a fixed small degree.** In this subsection $4 \le n \le 13$ is fixed. We begin with the Schoenberg bound in (2.3) $V_0 := \frac{n-2}{n-1}$. If $t_* := 1 - (n-1)\Delta_+ > 0$, define

$$D_+^2 = \frac{\Delta_+^2}{t_*}, \qquad V_0 \leftarrow \min\left\{\frac{n-2}{n-1}, 1 + \frac{\Delta_+}{t_*} - a_-^2\right\}.$$

Indeed, Lemma 6.6 gives

$$|a - \mu|^2 \le \frac{(1-s)^2}{1 - (n-1)(1-s)} \le D_+^2,$$

and

$$V_* \le 1 + \frac{1-s}{1-(n-1)(1-s)} - a^2 \le 1 + \frac{\Delta_+}{t_*} - a_-^2.$$

If $t_* \le 0$, the quantity D_+^2 is left unspecified and we retain only the original bound $V_0 = (n-2)/(n-1)$. A negative value of V_0 excludes the box immediately.

For the root-product estimates, put

$$B_+ = \frac{n}{n-1}(1 + a_+^2), \qquad C_+ = \frac{n}{(n-1)^2}(n - 1 - a_-^2)_+,$$
$$g_- = \max\left\{0,\, a_- - \frac{1 + a_+^2}{2} r_+,\, \frac{1 + a_-^2}{2} r_- - a_+\right\},$$

and define

$$\gamma_- = \begin{cases} \dfrac{n-1}{3}\left(\dfrac{n}{n-1}\right)^2 \dfrac{g_-^2}{C_+}, & C_+ > 0, \\ 0, & C_+ = 0. \end{cases}$$

Finally, set

$$L = \frac{n^2(1 + \gamma_-)}{B_+^{n-1}}.$$

By (6.6),

$$B_0 \le B_+, \qquad C_0 \le C_+,$$

and

$$\left|A_0 - \frac{B_0 \mu}{2}\right| = \frac{n}{n-1}\left|a - \frac{1 + a^2}{2}\mu\right| \ge \frac{n}{n-1} g_-.$$

Lemma 6.3 therefore gives

$$L \le \prod_{j=1}^{n-1} |q_j|^2.$$

If $C_0 = 0$, the covariance gap in Lemma 6.3 vanishes. Thus no division by zero occurs: a box containing admissible data with $C_0 = 0$ cannot simultaneously produce a strictly positive lower bound g_-.

Next, define

$$d_s = \max\{0, a_- s_- - r_+, r_- - a_+ s_+\}.$$

Since $|\mu - as| \ge |r - as| \ge d_s$, Lemma 6.4 gives the following refinement. If $s_+ V_0 > 0$, set

$$\lambda_- = \frac{(n-1)d_s^2}{2s_+ V_0};$$

otherwise set $\lambda_- = 0$. Then

$$\prod_{j=1}^{n-1} |q_j|^2 \le s_+^{n-1} e^{-\lambda_-}. \tag{A.1}$$

When $V_0 = 0$, the choice $\lambda_- = 0$ uses only the arithmetic–geometric mean bound and therefore involves no division by V_0.

Suppose now that $0 < L \le s_+^{n-1}$. The checker chooses η by 30 exact bisections on $[0, s_+]$, retaining a lower endpoint for which (6.20) holds with s_+ in place of s. Whenever this lower endpoint is positive, Lemma 6.7 supplies k, z, M, T. Let $M_+ \ge M$ be an upward-rounded upper bound. We then refine

$$D_+^2 \leftarrow \min\left\{D_+^2, \left(\frac{M_+}{n-1}\right)^2\right\}, \qquad V = \min\{V_0, T - a_-^2\}. \tag{A.2}$$

If D_+^2 was previously unspecified, the first minimum in (A.2) is interpreted as the newly available bound $(M_+/(n-1))^2$. If the packing refinement is unavailable, we simply set $V = V_0$.

The product loss (A.1) continues to use the original quantity V_0, rather than the subsequently refined V. Thus the order of the preceding refinements introduces no circularity.

For later use, define the positive Taylor polynomial

$$T_{32}(x) := \sum_{j=0}^{32} \frac{x^j}{j!}, \qquad x \ge 0. \tag{A.3}$$

Then

$$T_{32}(x) \le e^x.$$

For clarity, Table 2 lists the elementary exclusion tests used for a fixed degree. The tests involving only the original Schoenberg bound may use V_0, whereas `moment_A_packed` and `moment_B_packed` use the refined value V. Every strict inequality displayed in the table excludes the entire box.

The test `product_radial` follows from (6.16). Indeed, the function $x \longmapsto \frac{1-x}{\sqrt{x}}$ is decreasing on $(0, 1]$, and $L \le \prod_{j=1}^{n-1} |q_j|^2$. The test `finite_radial` is relevant only when

$$1 - (n-1)\Delta_+ > 0.$$

Finally, `near_uniform` is exactly Lemma 7.6; it is not a numerical approximation to that lemma.

For the remaining tests assume $V \ge 0, v_+ \ge 0$, and define

$$H = \max\left\{r_-^2 + (1-V)(1-s_+),\ 1 - (a_+^2 + V)v_+\right\}, \qquad A = a_-^2 s_-.$$

If D_+^2 is available, include

$$a_-^2(1 - s_+) + r_-^2 - D_+^2$$

TABLE 2. Elementary exclusion tests for a fixed degree.

Identifier	Sufficient condition for exclusion
`degree_bound`	$a_-^2 > n-1$
`variance_nonnegative`	$v_+ < 0$
`inverse_moment`	$(a_+^2 + V_0)s_+ < 1$
`moment_A`	$d_a^2 > (a_+^2 + V_0 - 1)_+(1 - s_-)$
`moment_B`	$d_h^2 > V_0 v_+$
`product`	$(B_+ s_+)^{n-1} < n^2$
`finite_radial`	$d_a^2(1-(n-1)\Delta_+) > \Delta_+^2$
`product_stable`	$(B_+ s_+)^{n-1} < n^2 T_{32}(\lambda_-)$
`product_root_stable`	$(B_+ s_+)^{n-1} < n^2(1+\gamma_-)T_{32}(\lambda_-)$
`product_radial`	$L > 1$, or $0 < L \le 1$ and $(n-1)^2 d_a^2 L > (1-L)^2$
`product_logcap`	$(n-1)d_a > M_+$ when M_+ is available
`critical_variance_negative`	$V_0 < 0$ or $V < 0$
`moment_A_packed`	$d_a^2 > (a_+^2 + V - 1)_+(1 - s_-)$
`moment_B_packed`	$d_h^2 > V v_+$
`near_uniform`	$r_- \ge 99999/100000$

as a third entry in the maximum defining H. Lemma 3.1 then gives

$$\rho \ge H.$$

The third entry follows from the identity

$$\rho = a^2(1-s) + r^2 - |a-\mu|^2.$$

Since $\beta_s(t) = 1 - \rho t - a^2 s\, t(1-t)$, we obtain the box-wise majorant

$$\beta_s(t) \le b(t) := 1 - Ht - At(1-t).$$

If $H > 1$, or if b has a negative minimum on $[0,1]$, the box is excluded; this is the test `beta_negative`. Otherwise the checker requires

$$0 \le H \le 1, \qquad b(t) \ge 0 \quad (0 \le t \le 1).$$

When $A > 0$, the only possible interior minimum occurs at $t = \frac{H+A}{2A}$, and its value is $1 - \frac{(H+A)^2}{4A}$. Thus this check is entirely rational.

Define

$$\kappa^2 = \min\{1-H, Vs_+\}, \qquad d^2 = \max\left\{0,\, \min\{Vv_+, 1-H-a_-^2 v_-\}\right\}, \qquad \ell(t) = 1 - t + \kappa t. \tag{A.4}$$

The first definition ensures (4.5). For the second, recall that $|1-a\mu|^2 \le Vv$ and $|1-a\mu|^2 = \beta_s(1) - a^2 v$. Hence, on an admissible box,

$$|1-a\mu|^2 \le \min\{Vv_+, 1-H-a_-^2 v_-\}.$$

Both quantities inside this minimum are then nonnegative, so the outer maximum with 0 does not weaken the asserted upper bound.

Let Φ_+ denote the maximum of Φ_{n-1} over $[a_-, a_+] \cap [0, \sqrt{n-1}]$. A box for which this intersection is empty has already been excluded by the degree bound. By Lemma 3.3, the maximum is attained at 1 if $1 \in [a_-, a_+]$, at a_+ if $a_+ < 1$, and at a_- if $a_- > 1$. Thus no irrational interval endpoint is needed.

Combining (3.7) and (A.1), and using $e^{\lambda_-/2} \ge T_{32}(\lambda_-/2)$, gives the interval upper bound

$$\mathcal{C}_+ = \frac{\Phi_+ r_+ s_+^{(n-1)/2}}{T_{32}(\lambda_-/2)} \ge |a\mu J|.$$

A box entirely outside $a^2 \le n-1$ has already been excluded, so every evaluation of Φ_+ has a nonnegative base.

The tests `direct` and `direct_retained` evaluate (4.16), using (4.6) alone or together with (4.7). The tests `center_absolute` and `center_ratio` evaluate (4.17) and (4.18); these tests are used only for $n \geq 5$. All integrals are bounded by (4.23) and (4.24), with the constants in (4.1).

For a mesh interval $I = [l, u]$, define

$$d_I := \max\{0, 1 - a_+ r_+ u, a_- r_- l - 1\}.$$

Then

$$|1 - a\mu t| \geq d_I \qquad (t \in I)$$

for every admissible point in the box. The nonsingular estimate (4.11) is always retained; the estimate (4.10) is used only when $d_I > 0$.

Whenever a centered integral test is evaluated, let $\mathcal{I}_{2,+}$ denote the rigorous upper bound, obtained from the chord estimates, for the corresponding integral $\int_0^1 t^2 \mathcal{U}(t)\,\mathrm{d}t$. We then write

$$K_+ := n a_+^3 r_+ \mathcal{I}_{2,+}.$$

By (4.15),

$$K \leq K_+.$$

Thus K_+ is simply the box-wise upper bound for the quantity K appearing in Lemma 4.3.

The test `center_coeff` uses (7.5) with $\mathcal{C}_+$, the value d from (A.4), and upper endpoints in every nonnegative coefficient.

The final test `center_positive` uses Lemma 7.5 in the form (7.19). It is used only when

$$r_- > 0, \qquad h_+ := a_+ r_+ \leq 1.$$

Set $h_- := a_- r_-$. If the packing refinement above is unavailable, put $\eta = 0$. Otherwise η denotes the positive lower endpoint produced by the packing bisection. Define

$$t_0 = \max\{0, \eta, 1 - (n-1)(1 - s_-)\},$$

$$R_- = \max\left\{-1,\ (n-1)r_- - (n-2),\ \frac{(n-1)^2 r_-^2 + t_0 - (n-2)^2}{2(n-1)r_-}\right\},$$

$$W_+ = \min\left\{(n-2)(1-r_-)^2,\ (1-r_-)(r_+ - R_-)\right\},$$

$$L_2 = \frac{n-1}{2}(v_- - 2W_+).$$

Each of the three lower bounds in (7.11) is nondecreasing in $r = |\mu| > 0$ when $0 \leq t_0 \leq 1$ and $n \geq 4$. Lemma 7.4 therefore gives

$$W \leq W_+.$$

The checker invokes this test only when $L_2 > 0$.

Finally set

$$x_- = \frac{H + a_-^2 s_-}{2a_+}, \qquad \varepsilon = \sqrt{\max\{0, 2(1 - x_-/r_+)\}}, \qquad K_- = n h_- I_{2,n-3}(1 - h_+).$$

The identity $2ax = \rho + a^2 s$ gives $x \geq x_-$. Since $|e^{i\theta} - 1|^2 = 2\left(1 - \frac{x}{r}\right)$, we obtain $|e^{i\theta} - 1| \leq \varepsilon$. Moreover,

$$K_2(a|\mu|) = n a|\mu| I_{2,n-3}(1 - a|\mu|) \geq n h_- I_{2,n-3}(1 - h_+) = K_-,$$

so that

$$0 \leq K_- \leq K_2(a|\mu|).$$

In (7.18), h_0 is replaced by the box upper bound h_+ when estimating E_2 from above. In (7.19), the favorable quadratic term is bounded from above by $-a_-^2 K_- L_2$. Because this quantity is subtracted, it is rounded in the direction appropriate for a lower bound on the positive factor $a^2 K_- L_2$, as explained in Appendix B.

A.2. **Several consecutive degrees in one block.** We now consider a degree block

$$n-1 \in [M,N], \qquad 13 \le M \le N.$$

The block checker uses only Lemma 3.1, Lemma 3.3, and the integral estimates of Section 4. Set $V = 1-\frac{1}{N}, v_+ = s_+ - r_-^2$, and

$$H = \max\left\{r_-^2 + \frac{1-s_+}{N},\, 1-(a_+^2+V)v_+\right\}, \qquad A = a_-^2 s_-.$$

Since

$$V_* \le 1 - \frac{1}{n-1} \le 1 - \frac{1}{N} = V,$$

these are valid uniform bounds throughout the degree block.

Use $b(t) = 1 - Ht - At(1-t), \kappa^2 = \min\{1-H, Vs_+\}$, together with the same nonnegativity checks as in Appendix A.1. The elementary tests are the degree bound, the nonnegative-variance test, the two covariance tests, the inverse-moment test, the negative-b test, and the product test described below.

Lemma 3.3 gives the uniform bound

$$|a\mu J| \le \mathcal{C}_+ := \max_{\substack{a\in[a_-,a_+]\\ a^2\le N}} \Phi_N(a)\, r_+ s_+^{M/2}.$$

The maximum of Φ_N is evaluated by the same endpoint rule used in Appendix A.1.

For the elementary product test, put

$$B = \frac{M+1}{M}(1+a_+^2)s_+.$$

If $B \le 1$, then the actual base in (3.8) is at most 1, whereas (3.8) requires its positive power to be at least $n^2 > 1$. Thus $B \le 1$ already excludes the box.

If $B > 1$, then increasing the exponent gives an upper bound, and the condition

$$B^N < (M+1)^2$$

again contradicts (3.8). This is the block product test.

For the direct integral estimate, we use the coefficients

$$\min\{1, (5/3)s_+\}, \qquad \min\{1, 3s_+\},$$

the powers

$$b^{(M-1)/2}, \qquad \ell^{M-1},$$

and the prefactor Na_+^2. The two endpoint terms in (4.16) are bounded by

$$\kappa^M, \qquad \frac{N}{N+1}\mathcal{C}_+.$$

These are valid upper bounds because

$$0 \le b, \ell, \kappa \le 1.$$

For `direct_retained`, let $I = [l,u]$ be a mesh interval and set $\delta_I := (1-a_+u)_+$. For every admissible point and every $t \in I$,

$$|1-atq_j| \ge (1-at|q_j|)_+ \ge \delta_I, \qquad 1 \le j \le n-1.$$

Consequently,

$$b(t) \ge \delta_I^2, \qquad \ell(t) \ge \delta_I.$$

For fixed t, the functions

$$\frac{(n-1)b(t)-\delta_I^2}{n-2}, \qquad \frac{(n-1)\ell(t)-\delta_I}{n-2}$$

are nonincreasing in $n-1$. We therefore enlarge their bases by setting $n-1=M$, and write

$$B_I(t) := \left(\frac{Mb(t)-\delta_I^2}{M-1}\right)_+, \qquad L_I(t) := \left(\frac{M\ell(t)-\delta_I}{M-1}\right)_+.$$

These enlarged bases need not be at most 1. The corresponding powers are therefore bounded by

$$\max\{B_I(t)^{(M-1)/2}, B_I(t)^{(N-1)/2}\}, \qquad \max\{L_I(t)^{M-1}, L_I(t)^{N-1}\}. \tag{A.5}$$

Both expressions are convex functions of t: the bases are nonnegative and convex, all exponents are at least one, and taking the maximum preserves convexity. We apply (4.23) to these two majorants separately.

For the centered integral estimates, the checker uses the four uniform coefficients

$$\frac{5N}{6d_I}, \qquad \frac{3N}{2d_I}, \qquad \frac{3N(N-1)}{2}, \qquad \frac{9N(N-1)}{2}.$$

The first two are omitted when $d_I=0$. The corresponding powers are

$$b^{(M-1)/2}, \qquad \ell^{M-1}, \qquad b^{(M-2)/2}, \qquad \ell^{M-2}.$$

In the definition of K, the prefactor satisfies

$$na^3r \le (N+1)a_+^3 r_+.$$

Thus K_+ denotes the upper bound obtained by multiplying $(N+1)a_+^3r_+$ by the rigorous chord bound for the relevant integral $\int t^2\mathcal{U}(t)\,\mathrm{d}t$.

For (4.17), we use

$$(Vv)^{n/2} \le (Vv_+)^{(M+1)/2}.$$

For (4.18), we use

$$(1+r_+)\left\{V^{(M+1)/2}(1-r_-^2)^{(M-1)/2}+K_+\right\}.$$

All bases in these centered powers lie in $[0,1]$, so decreasing the exponent enlarges the expression. The powers in (A.5) are the deliberate exception, because their bases are not known a priori to be at most 1.

Appendix B. Exact arithmetic and reproducibility

B.1. **Certificates and coverage.** The two certificate files `certificate_n4_to_n13.json` and `certificate_n14_to_n100000.json` encode the partitions used in Proposition 8.2.

A degree block records an inclusive range $[M,N]$ for $n-1$, a positive mesh-refinement parameter, and a finite list of leaf paths together with their test identifiers. The initial parameter box is (8.5).

Each pair of characters in a path records a coordinate index in $\{0,1,2\}$ and a child index in $\{0,1\}$. The corresponding coordinate interval is bisected at its exact midpoint. Both children are represented by closed intervals, so adjacent leaves may share a boundary but have disjoint interiors.

The checker inserts all paths into a tree. It rejects repeated leaves, a leaf with descendants, conflicting split coordinates, or an internal node missing either child. All endpoints are reconstructed as exact rational numbers. Induction on this finite tree proves that the leaves cover the initial box. The checker also verifies that consecutive degree blocks have neither gaps nor overlaps. Thus no information from the floating-point certificate generator is accepted merely because it appears in a certificate.

For a block beginning at $n-1=M$, set

$$L_{\mathrm{mesh}} = \max\{4, \lceil \log_2 M\rceil + 2\}.$$

The mesh first divides $[0,2^{-L_{\mathrm{mesh}}}]$ into q equal subintervals, and then divides each interval

$$[2^{-j}, 2^{-j+1}], \qquad j = L_{\mathrm{mesh}}, \dots, 1,$$

into q equal subintervals. The small-degree certificates use $q = 32$, the degree-14 block uses $q = 16$, and all remaining blocks use $q = 4$. The checker verifies directly that the resulting rational mesh points increase from 0 to 1.

The integral bounds are exactly (4.23) and (4.24); no floating-point quadrature is used in the verification.

B.2. **Arithmetic operations.** Fix $S = 2^\nu$, where the primary verification uses $\nu = 100$. Here ν denotes the bit precision and is unrelated to the majorant $b(t)$ used earlier.

A nonnegative integer U represents the rigorous upper bound U/S. For a nonnegative rational number p/q, its upward enclosure is $\left\lceil \frac{Sp}{q} \right\rceil$. Multiplication of upper bounds is implemented by

$$(U, V) \longmapsto \left\lceil \frac{UV}{S} \right\rceil, \qquad \left(\frac{p}{q}, U\right) \longmapsto \left\lceil \frac{pU}{q} \right\rceil. \tag{B.1}$$

For a square root, if U/S is the current upper bound, the checker chooses the smallest nonnegative integer W satisfying $W^2 \geq US$. Then

$$\frac{W}{S} \geq \sqrt{\frac{U}{S}}.$$

The integer W is obtained from the integer square root of US, with one added unless US is a perfect square.

Integer powers are evaluated by repeated squaring using (B.1). For a half-integer power $x^{k+1/2}$, the checker multiplies an upward enclosure of x^k by an upward enclosure of $\sqrt{x}$, again rounding upward.

The favorable negative term in (7.19) requires the opposite rounding direction. If $c \geq 0$ is a rational lower bound for the positive quantity $a^2 K_- L_2$, the checker subtracts $\lfloor Sc \rfloor$, rather than $\lceil Sc \rceil$, from the upper-bound numerator. Thus directed rounding can never make the final upper bound artificially small.

Exponential factors are handled using the positive Taylor polynomial T_{32} from (A.3); no floating-point exponential or logarithm is used in an acceptance decision.

The interpretation of every stored integer as a rigorous upper or lower bound is preserved inductively by these operations. A main numerical test is accepted only when its final upper-bound numerator U satisfies $U < S$. Thus the positive margins in (8.6) represent strict mathematical inequalities, not numerical tolerances. The elementary exclusions are comparisons of exact rational numbers. Timing information in the verification logs plays no role in acceptance.

B.3. **Reproduction.** The essential supplementary files are `certificate_n4_to_n13.json`, `certificate_n14_to_n100000.json`, `verify_low.py`, `verify_mid.py`, and `verify_all.py`.

The checker `verify_low.py` treats $4 \leq n \leq 13$ using Appendix A.1, while `verify_mid.py` treats $14 \leq n \leq 100000$ using Appendix A.2. The former imports the integer-arithmetic routines and mesh construction from the latter, but neither checker imports a certificate generator.

The wrapper `verify_all.py` invokes both checkers, verifies their degree endpoints, and records the combined result. The arithmetic self-tests are contained in `self_test.py`.

Use Python 3.10 or later, without optimization flags. From the root of the repository at the pinned commit, run

```
cd supplement
python3 self_test.py
python3 verify_all.py --bits 100
python3 verify_all.py --bits 160
```

Only the Python standard library is required. The optional certificate generators are not used by these commands. The optional certificate generators require NumPy, but they are not part of the acceptance procedure.

The two certificate files are identified by the following SHA–256 digests; line breaks below are not part of the digests.

```
certificate_n4_to_n13.json
4334dd70fe5317040124115239f92ecc
4e220cbf6c3576407396c3ea9df19f8b

certificate_n14_to_n100000.json
f9131604a764777836afed040bd73b1f
8ca24c5696022460b5547cb509af766d
```

For the first certificate, the numbers of leaves in degrees $4, \dots, 13$ are

$$532,\ 1103,\ 1594,\ 2540,\ 1753,\ 1420,\ 1164,\ 1002,\ 904,\ 809,$$

respectively. Ten leaves, five in degree 4 and five in degree 5, invoke Lemma 7.6. Every other leaf is checked by one of the exact rational or integer tests specified in Appendix A.

The complete runs at 100 and 160 bits verify the same partitions. They are checks at two different precisions, rather than two independent implementations of the full argument. Neither run is a proof-assistant formalization of the analytic lemmas. The arithmetic self-tests verify elementary identities, selected analytic constants, and rejection of malformed or deliberately false certificates; these tests supplement, but do not replace, any mathematical argument.

B.4. **Elementary analytic constants.** For completeness, the exponential comparisons used in Proposition 5.1 can also be verified entirely by rational arithmetic. If $x \geq 0$ and $x < k+2$, then

$$\sum_{j=0}^{k} \frac{x^j}{j!} \leq e^x \leq \sum_{j=0}^{k} \frac{x^j}{j!} + \frac{x^{k+1}}{(k+1)!} \frac{1}{1 - x/(k+2)}. \tag{B.2}$$

The upper bound follows because, beginning with the $(k+1)$-st term, the ratio of successive terms in the exponential series is at most $x/(k+2) < 1$. Hence the remaining tail is bounded by a geometric series.

Formula (B.2) with $k = 20$ verifies the four exponential comparisons used in Step 2 of Proposition 5.1. It also yields

$$e < \frac{11}{4} < \frac{25}{9},$$

which implies the uniform bounds following (4.1). The remaining estimates in Proposition 5.1 use only the displayed rational inequalities,

$$1 - u \leq e^{-u} \qquad (u \geq 0),$$

and

$$e^x \geq \frac{x^6}{720} \qquad (x \geq 0).$$

Thus the large-degree argument is analytic throughout its stated range; none of its estimates is obtained by extrapolating the finite verification.

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