A SHARP LOWER BOUND FOR PERMANENTS OF NEARLY DECOMPOSABLE DOUBLY STOCHASTIC MATRICES

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Abstract. We prove Foregger's conjecture: every nearly decomposable doubly stochastic matrix A of order n satisfies $\operatorname{per} A \geq 2^{1-n}$. The constant is best possible for every n, with equality for $\frac{1}{2}(I+P)$ when P is the permutation matrix of an n-cycle. We establish a stronger inequality for every elementary bipartite support graph. It is expressed through an entropy potential and a correction term indexed by removable edges; the correction vanishes for minimal elementary graphs and yields the stated matrix bound.

1. Introduction

Let S_n denote the symmetric group on $\{1,\dots,n\}$. For $A=(a_{ij})\in\mathbb{R}^{n\times n}$, its permanent is defined by

$$\operatorname{per} A=\sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i,\sigma(i)}.$$

A matrix is *doubly stochastic* if it is nonnegative and all of its row and column sums equal 1. Although the permanent differs from the determinant only in the absence of signs, its extremal behavior is markedly subtler. The minimization problem goes back to van der Waerden [23]. The celebrated van der Waerden conjecture asserted that every doubly stochastic matrix A satisfies

$$\operatorname{per} A\geq \frac{n!}{n^n},$$

with equality when $A=\frac{1}{n}J_n$, where J_n denotes the $n\times n$ all-ones matrix. It was proved independently by Egorychev [9] and Falikman [10]. This is the optimal bound without restrictions on the zero pattern. When the support is sparse but remains irreducible in the appropriate combinatorial sense, substantially stronger lower bounds become possible.

A nonnegative $n\times n$ matrix is *fully indecomposable* if it contains no $r\times(n-r)$ zero submatrix for $1\leq r\leq n-1$. It is *nearly decomposable* if it is fully indecomposable but ceases to be so whenever any positive entry is replaced by zero. This notion was introduced by Sinkhorn and Knopp [22]. Its structural and constructive theory was subsequently developed by Hartfiel [13, 14], while Brualdi and Hedrick [3] gave a unified treatment alongside the parallel theory of nearly reducible matrices. In graph theory, the support graphs arising from nearly decomposable matrices are precisely the minimal elementary bipartite graphs studied by Lovász and Plummer

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[18]. See Lovász [16] for the related ear-decomposition viewpoint, Brualdi [1, Section 1.3 and Chapters 8–9] for the matrix and polyhedral background, and Lovász and Plummer [19, Chapters 4–5] for the matching-theoretic background.

In 1980, Foregger [11] proved the bound below for $2 \le n \le 9$ and formulated the following conjecture for arbitrary order. Cheon and Wanless later recorded it as Conjecture 15 in their survey [6, p. 326], and Zhan included it as Conjecture 13 in his list of open problems [25].

Foregger's conjecture is the following.

Conjecture 1.1. *Let $n \ge 2$, and let $A \in \mathbb{R}^{n\times n}$ be a nearly decomposable doubly stochastic matrix. Then*

$$\operatorname{per} A \ge 2^{1-n}.$$

The constant in Conjecture 1.1 is attained by the matrices $\frac{1}{2}(I+P)$, where P is the permutation matrix of an n-cycle. Thus the conjecture asks for the exact minimum on the faces determined by minimal elementary support graphs. It is a genuine strengthening of the van der Waerden bound, since $\frac{n!}{n^n} < 2^{1-n}$ for $n \ge 3$.

We prove the conjecture.

Theorem 1.2. *Let $n \ge 2$, and let $A \in \mathbb{R}^{n\times n}$ be a nearly decomposable doubly stochastic matrix. Then*

$$\operatorname{per} A \ge 2^{1-n}.$$

The proof gives a stronger statement for arbitrary elementary support graphs. If G is an elementary labeled bipartite multigraph of bipartite order k, and x is a strictly positive fractional perfect matching, define

$$\Phi(G,x) = S_G(x) - H_{\max}(G,x)$$

as in Section 2. We prove

$$\Phi(G,x) \le (k-1)\log 2 + \sum_{e\in\mathcal{R}(G)} \big(\log 2 - h(x_e)\big). \tag{1.1}$$

For a minimal elementary graph the correction term is empty. Combined with the entropy reduction in Lemma 2.4, this implies Theorem 1.2. The sharpness examples are given in the proof of that theorem.

1.1. **Proof strategy.** The permanent is first expressed through the maximum entropy of a distribution on perfect matchings with prescribed edge marginals. A degree-sensitive coefficient–capacity estimate of Gurvits [12], stated in our notation in Lemma 3.1, bounds the resulting entropy potential. The degree terms are converted into binary-entropy deficits. Tight-cut splicing then gives exact additivity of Φ, and induction over tight-cut contractions proves (1.1); the quadrilateral case is handled by the explicit calculation in Lemma 4.5 and the bicontraction argument in Lemma 4.6.

1.2. **Organization.** Section 2 translates the matrix problem into labeled matching-covered bipartite graphs and establishes the entropy reduction. Section 3 states the capacity estimate and derives the rooted bound used in the induction. Section 4 proves exact additivity and the tight-cut induction. Section 5 then deduces the permanent bound and proves sharpness.

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2. Support graphs and entropy

We begin with the precise graph formulation. Parallel edges will be needed after tight-cut contractions, and they are always regarded as distinct labeled edges.

All graphs in this paper are finite, undirected, and loopless. A *matching* is a set of pairwise vertex-disjoint labeled edges, and it is *perfect* if it covers every vertex. For a vertex v, the labeled degree $d_G(v)$ counts parallel edge labels with multiplicity; this may exceed the number of distinct neighbors of v. For $F \subseteq E(G)$, the graph $G - F$ retains all vertices and deletes precisely the labels in F. A *parallel class* (also called an *edge bundle*) is the set of all labels with the same endpoint pair, and the *underlying simple graph* is obtained by replacing every nonempty bundle by one edge. These conventions agree with the standard theory of perfect matchings and matching-covered graphs; see [19, Chapters 4–5] and [20, pp. 3–40]. We spell them out because the distinction between labeled degree and number of distinct neighbors is used in Section 3.

2.1. **Matching-covered support graphs.** Let $G = (L \cup R, E)$ be a bipartite multigraph with $|L| = |R| = k$. The *bipartite order* of G is k, the common cardinality of its two color classes. A labeled edge is *admissible* if it belongs to a perfect matching. The graph is *elementary*, or *matching-covered*, if it is connected and every labeled edge is admissible. A labeled edge e is *removable* if $G - e$ is elementary, and $\mathcal{R}(G)$ denotes the set of all removable labeled edges. An admissible edge is also called *allowed*. For connected bipartite graphs, the preceding definition of elementary is equivalent to the usual elementary bipartite-graph definition; see [19, Theorem 4.1.1 and pp. 140–141]. The notions of matching-covered graphs and removable edges are treated systematically in [20, pp. 23–40 and 155–182]; throughout this paper these terms are applied to labeled edges, so deleting one of several parallel labels need not delete its endpoint pair. An elementary graph is *minimal elementary* if none of its labeled edges is removable, equivalently if $\mathcal{R}(G) = \varnothing$; this is the class studied in [18].

A vector $x = (x_e)_{e \in E}$ is a *strictly positive fractional perfect matching* if

$$x_e > 0 \quad (e \in E), \qquad \sum_{e \ni v} x_e = 1 \quad (v \in L \cup R). \tag{2.1}$$

By the bipartite perfect-matching polytope theorem [19, Theorem 7.1.2] (see also [20, pp. 111–137]), x is a convex combination of incidence vectors of labeled perfect matchings, where the incidence vector $\mathbf{1}_M \in \{0,1\}^E$ has $(\mathbf{1}_M)_e = 1$ precisely when $e \in M$. For the matrix-polytope setting and its permanent applications, see also [2]. Parallel labels are simply treated as distinct edges; equivalently, one may split the weight of a selected endpoint pair among its labels. Thus the theorem gives a distribution on labeled perfect matchings with edge marginals x_e.

Given a nonnegative matrix $A = (a_{ij})$, its support graph $G(A)$ is the bipartite graph with row-vertex class $L = \{1, \dots, n\}$, column-vertex class $R = \{1, \dots, n\}$, and an edge ij exactly when $a_{ij} > 0$. For a nonnegative square matrix, full indecomposability is equivalent to the strict Hall condition on every nonempty proper set of row vertices, and hence to the support graph being elementary; see [15, Corollary 3.5], [1, Section 1.3], and [19, Theorem 4.1.1 and pp. 140–141]. Replacing a positive entry by zero deletes exactly the corresponding support edge, so the same equivalence identifies nearly decomposable matrices with minimal elementary supports; see also [18].

Lemma 2.1. *A nonnegative square matrix A of order $n \geq 2$ is fully indecomposable if and only if $G(A)$ is elementary.*

Proof. For a nonnegative square matrix, full indecomposability is equivalent to the strict Hall condition for every nonempty proper subset of one color class. For a connected bipartite graph, that condition is equivalent to every edge belonging to a perfect matching. This proves the assertion. □

The matrix formulation of near decomposability is the following corollary.

Corollary 2.2. *If A is doubly stochastic, then A is nearly decomposable if and only if $G(A)$ is elementary and $\mathcal{R}(G(A)) = \varnothing$.*

Proof. Replacing a positive matrix entry by zero deletes exactly the corresponding support edge. By Lemma 2.1, near decomposability is therefore equivalent to elementarity together with the failure of elementarity after every edge deletion. This is equivalent to $\mathcal{R}(G(A)) = \varnothing$. □

This equivalence allows us to work entirely with perfect matchings. The weights of a doubly stochastic matrix satisfy (2.1) on its support graph.

2.2. **The maximum-entropy potential.** Let $\mathcal{M}(G)$ be the finite set of labeled perfect matchings of an elementary bipartite multigraph G. For a probability vector $p = (p_\omega)_{\omega \in \Omega}$ on a finite set, write

$$H(p) = -\sum_{\omega \in \Omega} p_\omega \log p_\omega.$$

We use the standard finite-alphabet notions of Shannon entropy, conditional entropy, and relative entropy. In particular, we will use the chain rule, entropy subadditivity, the fact that conditioning cannot increase entropy, and nonnegativity of relative entropy; see [7, Chapter 2]. All logarithms are natural, so entropy is measured in nats. For a strictly positive fractional perfect matching x, let $\mathcal{P}(G, x)$ be the set of probability distributions μ on $\mathcal{M}(G)$ satisfying If M is sampled from μ, the constraints are

$$\Pr_\mu(e \in M) = x_e \qquad (e \in E). \tag{2.2}$$

The set specified by (2.2) is nonempty by the perfect-matching polytope theorem. With the convention $0 \log 0 = 0$, define

$$H_{\max}(G,x) = \max_{\mu \in \mathcal{P}(G,x)} \left(- \sum_{M \in \mathcal{M}(G)} \mu(M) \log \mu(M) \right). \tag{2.3}$$

The maximum in (2.3) exists by compactness.

The sum of the local entropies on either color class is

$$S_G(x) = \sum_{u \in L} H\big((x_e)_{e \ni u}\big) = - \sum_{e \in E} x_e \log x_e. \tag{2.4}$$

The identity (2.4) also results from summing over R. We set

$$\Phi(G,x) = S_G(x) - H_{\max}(G,x). \tag{2.5}$$

It follows that $\Phi(G,x) \geq 0$. Indeed, let M be a random perfect matching with marginals x, and let E_u be the labeled edge of M incident with $u \in L$. The tuple $(E_u)_{u \in L}$ determines the matching, so entropy subadditivity and (2.4) give

$$H(M) \leq \sum_{u \in L} H(E_u) = S_G(x).$$

Only an upper bound on Φ will be needed.

The variational form of $H_{\max}$ will connect this potential to stable-polynomial capacity. For positive edge parameters $w = (w_e)_{e \in E}$, write

$$Z_G(w) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} w_e. \tag{2.6}$$

The finite Gibbs variational principle is reviewed in [24, Theorem 3.4]. The prescribed-marginal formulation, including the case in which the marginal polytope has a lower-dimensional affine hull, is stated explicitly in [21, Lemma 2.3 and Appendix A.1 of the full version]. We record the specialization needed here, with Z_G as in (2.6).

Lemma 2.3. *For every elementary labeled bipartite multigraph G and every strictly positive fractional perfect matching x of G,*

$$H_{\max}(G,x) = \inf_{w_e > 0} \left\{ \log Z_G(w) - \sum_{e \in E} x_e \log w_e \right\}. \tag{2.7}$$

Proof. The labeled bipartite perfect-matching polytope is described, in its affine hull, by the vertex equations together with the nonnegative coordinate inequalities. Because G is matching-covered, no edge coordinate vanishes throughout the polytope. Hence $x_e > 0$ for every edge label implies that x lies in the relative interior. Apply the prescribed-marginal Gibbs variational formula cited above to the sufficient statistic $M \mapsto \mathbf{1}_M$. On writing the exponential parameters as $\log w_e$, its log-partition function is $\log Z_G(w)$, and the resulting dual formula is exactly (2.7). □

Substituting $w = a$ into the right-hand side of (2.7) yields the one-sided Gibbs variational inequality [7, Section 2.7]; equivalently, this is the weighted arithmetic–geometric mean inequality. The resulting permanent bound is recorded next.

Lemma 2.4. *Let $A = (a_{ij})$ be doubly stochastic, and suppose that its support graph $G = G(A)$ is elementary. Regard the positive entries of A as the edge-weight vector $a = (a_e)_{e\in E(G)}$, where $a_e = a_{ij}$ for the edge $e = ij$. Then*

$$\operatorname{per} A \geq \exp\bigl(-\Phi(G, a)\bigr). \tag{2.8}$$

Proof. Evaluate (2.7) at $w = a$. Since $Z_G(a) = \operatorname{per} A$ and $\sum_e a_e \log a_e = -S_G(a)$, we obtain

$$H_{\max}(G, a) \leq \log(\operatorname{per} A) + S_G(a).$$

Rearranging and using (2.5) gives $\log(\operatorname{per} A) \geq -\Phi(G, a)$, which is equivalent to (2.8). □

Accordingly, the matrix inequality will follow once the potential is controlled on minimal elementary support graphs.

3. A rooted form of Gurvits's coefficient–capacity estimate

The induction requires a capacity estimate in which one prescribed vertex incurs no degree loss. This asymmetry is already present in the ordered product in Gurvits's coefficient–capacity theorem: the first variable contributes no factor. We record the specialization needed below. For a finite index set T and a homogeneous polynomial $q \in \mathbb{R}[y_j : j \in T]$ of degree $|T|$ with nonnegative coefficients, define

$$\operatorname{Cap}_T(q) = \inf_{(y_j)_{j\in T}>0} \frac{q(y)}{\prod_{j\in T} y_j}. \tag{3.1}$$

For an integer $r \geq 2$, put

$$\beta_r = \left(\frac{r-1}{r}\right)^{r-1}, \qquad c_r = -\log \beta_r = (r-1)\log\frac{r}{r-1}. \tag{3.2}$$

We also set $\beta_1 = 1$ and $c_1 = 0$. The continuous extension $c(r) = (r-1)\log\bigl(r/(r-1)\bigr)$ is strictly increasing for $r > 1$, since

$$c'(r) = \log\frac{r}{r-1} - \frac{1}{r} > 0.$$

(The last inequality is $\log(1 + t) > t/(1 + t)$ with $t = 1/(r - 1)$.) Consequently, $(c_r)_{r\geq 1}$ is increasing and $(\beta_r)_{r\geq 1}$ is decreasing.

A form of Gurvits's coefficient–capacity estimate, due to Gurvits [12], will be used repeatedly. The result is stated here in the notation of this paper; We call a polynomial H-stable if it is nonzero whenever all variables have positive real parts. With this convention, the result takes the following form.

Lemma 3.1. *Let q be a homogeneous H-stable polynomial of degree k in variables $y_1, \dots, y_k$, with nonnegative coefficients, and let $d_i = \deg_{y_i} q$. If $[y_1 \cdots y_k]q$ denotes the coefficient of $y_1 \cdots y_k$, then*

$$[y_1 \cdots y_k]q \geq \operatorname{Cap}_{\{1,\dots,k\}}(q) \prod_{i=2}^{k} \beta_{\min\{i,d_i\}}. \tag{3.3}$$

We now apply Lemma 3.1 in the following rooted form. The variable corresponding to the contraction vertex will be placed first; this is why the cut edges receive no degree loss.

Lemma 3.2. *Let $G = (L \cup R, E)$ be an elementary labeled bipartite multigraph of bipartite order k, let $w_e > 0$, and fix a vertex $z \in R$. For $j \in R$, let ν_j be the number of* distinct *neighbors of j in L. Then*

$$Z_G(w) \geq \mathrm{Cap}_R(p_w) \prod_{j \in R \setminus \{z\}} \beta_{\nu_j}, \tag{3.4}$$

where

$$p_w(y) = \prod_{i \in L} \left(\sum_{\substack{e \in E \\ e \ni i}} w_e y_{\rho(e)} \right) \tag{3.5}$$

and $\rho(e)$ denotes the right endpoint of e.

Proof. The polynomial p_w defined by (3.5) is homogeneous of degree k in the k variables indexed by R, and the coefficient of $\prod_{j \in R} y_j$ is $Z_G(w)$. Moreover, $\deg_{y_j} p_w = \nu_j$: parallel labels joining the same two vertices do not increase the degree in y_j. If all variables have positive real parts, each factor has positive real part; hence p_w is H-stable.

Place y_z first in the ordered product in Lemma 3.1. The first variable incurs no loss, while a variable y_j, $j \neq z$, occupying position i, contributes at least β_{ν_j}, because

$$\beta_{\min\{i, \deg_{y_j} p_w\}} \geq \beta_{\deg_{y_j} p_w} = \beta_{\nu_j}.$$

Applying (3.3) in this ordering and using the monotonicity of β_r gives (3.4). □

The remaining capacity in (3.4) is controlled by the pointwise weighted AM–GM inequality. At each left vertex, the weights x_e sum to one; after multiplication, the right-vertex equations cancel $\prod_j y_j$. Equivalently, this is the log-sum inequality applied separately at each left vertex; see [7, Section 2.7].

Lemma 3.3. *Under the hypotheses of Lemma 3.2, let x be a strictly positive fractional perfect matching of G. Then*

$$\mathrm{Cap}_R(p_w) \geq \exp\left(\sum_{e \in E} x_e \log w_e + S_G(x) \right). \tag{3.6}$$

Proof. For every $i \in L$, weighted AM–GM and $\sum_{e \ni i} x_e = 1$ give

$$\sum_{e \ni i} w_e y_{\rho(e)} \geq \prod_{e \ni i} \left(\frac{w_e y_{\rho(e)}}{x_e} \right)^{x_e}.$$

Multiplying these inequalities over $i \in L$ and then using $\sum_{e \ni j} x_e = 1$ for every $j \in R$ yields

$$p_w(y) \geq \left(\prod_{j \in R} y_j \right) \exp\left(\sum_{e \in E} x_e \log w_e + S_G(x) \right).$$

Divide by $\prod_{j \in R} y_j$ and take the infimum in (3.1). □

Combine (3.4) with (3.6). Applying Lemma 2.3 then yields the corresponding estimate for Φ. Every loss is concentrated in the explicit degree costs c_{ν_j}, while the chosen root incurs none.

Proposition 3.4. *Let $G = (L \cup R, E)$ be an elementary labeled bipartite multigraph of bipartite order k, let x be a strictly positive fractional perfect matching of G, and fix $z \in R$. For $j \in R$, let ν_j denote the number of distinct neighbors of j. Then*

$$\Phi(G, x) \leq \sum_{j \in R \setminus \{z\}} c_{\nu_j}. \tag{3.7}$$

Proof. For every positive edge-weight vector w, Lemmas 3.2 and 3.3 give

$$\log Z_G(w) - \sum_{e \in E} x_e \log w_e \geq S_G(x) - \sum_{j \in R \setminus \{z\}} c_{\nu_j}.$$

Take the infimum over $w_e > 0$ and use (2.7). Subtracting the resulting lower bound on $H_{\max}(G, x)$ from $S_G(x)$ proves (3.7). □

To make the estimate compatible with edge deletion, we convert each degree term into binary entropy deficits. Define

$$h(t) = -t \log t - (1-t) \log(1-t), \qquad \delta(t) = \log 2 - h(t). \tag{3.8}$$

Thus $\delta(t) \geq 0$ for $0 \leq t \leq 1$, and $\delta(t) = \delta(1-t)$.

The capacity loss c_d is attached to a right vertex, whereas the induction must retain corrections edge by edge. This mismatch leads to the deficit δ: it is invariant under $t \leftrightarrow 1-t$, vanishes at the balanced weight, and is complementary to the binary entropy that appears at a quadrilateral brace. Concavity of the binary entropy [7, Chapter 2] gives $\sum_{\nu=1}^{d} h(x_\nu) \leq dh(1/d) = \log d + c_d$. Consequently, the desired conversion $c_d \leq \log 2 + \sum_\nu \delta(x_\nu)$ reduces exactly to $\log d + 2c_d \leq (d+1) \log 2$. It remains only to verify the scalar inequality that performs this conversion.

Lemma 3.5. *For every integer $d \geq 2$,*

$$\log d + 2c_d \leq (d+1) \log 2. \tag{3.9}$$

Proof. Equality holds in (3.9) for $d = 2$. For real $d \geq 3$, set $F(d) = (d+1) \log 2 - \log d - 2(d-1) \log(d/(d-1))$. Then $F(3) = \log(256/243) > 0$, and

$$F'(d) = \log 2 + \frac{1}{d} - 2 \log\left(1 + \frac{1}{d-1}\right) \geq \log 2 + \frac{1}{d} - \frac{2}{d-1} \geq 0.$$

The last expression is positive at $d = 3$ and has derivative $2/(d-1)^2 - 1/d^2 > 0$. This proves (3.9). □

The scalar estimate implies the following edgewise form.

Corollary 3.6. *If $x_1, \ldots, x_d > 0$ and $\sum_{\nu=1}^{d} x_\nu = 1$, then*

$$c_d \leq \log 2 + \sum_{\nu=1}^{d} \delta(x_\nu). \tag{3.10}$$

Proof. By Lemma 3.5 and concavity of h, $\sum_{\nu=1}^{d} h(x_\nu) \leq dh(1/d) = \log d + c_d$. Together with (3.9), this gives

$$\log 2 + \sum_{\nu=1}^{d} \delta(x_\nu) = (d+1) \log 2 - \sum_{\nu=1}^{d} h(x_\nu) \geq c_d,$$

which is (3.10). □

For the brace-producing contraction, every cut edge is incident with the contraction vertex. We therefore place that vertex first in the rooted estimate, so no cut edge receives a binary-entropy deficit. Combining this observation with Corollary 3.6 converts the degree costs in Proposition 3.4 into precisely the edgewise correction needed by the induction.

Proposition 3.7. *Let $G = (L \cup R, E)$ be an elementary labeled bipartite multigraph of bipartite order $k \geq 2$, let x be a strictly positive fractional perfect matching of G, and fix $z \in R$. Then*

$$\Phi(G, x) \leq (k-1)\log 2 + \sum_{\substack{e \in E \\ e \not\ni z}} \delta(x_e). \tag{3.11}$$

Proof. Let d_j be the labeled degree of $j \in R$. Since G is elementary and $k \geq 2$, every vertex has at least two incident labeled edges. Indeed, a vertex of labeled degree one forces its unique incident edge to occur in every perfect matching; no other edge at the opposite endpoint could then be admissible, and connectivity would force $k = 1$. Moreover, $\nu_j \leq d_j$, and the sequence c_r defined in (3.2) is increasing.

Apply Corollary 3.6, (3.10), to the incident edge weights at each $j \neq z$. Proposition 3.4, in the form (3.7), then gives

$$\Phi(G, x) \leq \sum_{j \neq z} c_{\nu_j} \leq \sum_{j \neq z} c_{d_j} \leq \sum_{j \neq z} \left(\log 2 + \sum_{e \ni j} \delta(x_e) \right),$$

which is exactly (3.11). □

4. Tight cuts and the entropy potential

We next prove the global potential inequality. Shores, tight cuts, cut contractions, and braces are used in their standard matching-covered-graph sense; see [19, Section 7.6, pp. 295–302], [5, Sections 1.1–1.2], and [20, pp. 57–109]. Every contraction below retains the original edge labels, so parallel edges are never merged. Tight-cut decomposition originates in the brick-and-brace theory of Edmonds, Lovász, and Pulleyblank [8]; Lovász proved the invariance of the resulting list of bricks and braces [17]. Only the local contraction and matching-splicing properties enter the argument.

4.1. Exact additivity across a tight cut.

Fix an elementary labeled bipartite multigraph $G = (L \cup R, E)$ of bipartite order k and a strictly positive fractional perfect matching x of G. For $X \subseteq V(G)$, let ∂X be the set of edges with exactly one endpoint in X. The cut ∂X is *tight* if every perfect matching of G contains exactly one edge of ∂X. It is *nontrivial* if both shores have more than one vertex. The sets X and $\overline{X} = V(G) \setminus X$ are the two *shores* of the cut; either shore of a tight cut is called a *tight shore*.

To *contract* a shore means to replace all its vertices by one contraction vertex, delete any resulting loops, and retain every edge crossing the cut with its original label; parallel labels are never identified.

Write $L_X = L \cap X$ and $R_X = R \cap X$. After interchanging the two shores or the two color classes if necessary, the standard orientation theorem for bipartite tight

cuts [5, Proposition 5] allows us to assume

$$|L_X| = |R_X| + 1$$

and that every cut edge runs from L_X to $R \setminus X$. Contracting the two shores separately therefore produces two elementary labeled bipartite multigraphs G_1 and G_2. If G_1, G_2, and G have bipartite orders k_1, k_2, and k, respectively, then

$$k_1 + k_2 = k + 1. \tag{4.1}$$

Moreover, nontriviality gives $2 \le k_i \le k - 1$ for $i = 1, 2$.

By the perfect-matching polytope theorem, x is a convex combination of perfect matchings. Tightness therefore gives

$$\sum_{e \in \partial X} x_e = 1. \tag{4.2}$$

The vector x consequently descends to fractional perfect matchings $x^{(1)}$ and $x^{(2)}$ on the contractions. By (4.2), the label Z of the unique cut edge in a random perfect matching has law $\Pr(Z = e) = x_e$, $e \in \partial X$.

Under the standard splicing correspondence [8, 17] and [5, Section 1.1], a perfect matching of G is a compatible pair of contraction matchings with the common cut-edge label Z. Conditioning on Z separates the two shores, and conditional independence given Z maximizes the entropy [7, Sections 2.2 and 2.5]. The term $H(Z)$ is counted with the same multiplicity in the local entropy and in the maximum perfect-matching entropy, so it cancels from their difference. This is the reason for introducing Φ, and it yields exact additivity.

Lemma 4.1. *Let G be an elementary labeled bipartite multigraph, let x be a strictly positive fractional perfect matching of G, and let G_1, G_2 be the two contractions of a tight cut. If $x^{(i)}$ is the fractional perfect matching induced by x on G_i, then*

$$\Phi(G, x) = \Phi(G_1, x^{(1)}) + \Phi(G_2, x^{(2)}). \tag{4.3}$$

Proof. Under this correspondence, a perfect matching of G uniquely determines a pair of perfect matchings M_1, M_2 of the two contractions that use the same labeled cut edge Z. For any distribution in $\mathcal{P}(G, x)$, the induced distributions of M_i belong to $\mathcal{P}(G_i, x^{(i)})$. Since Z is a function of each M_i, conditional subadditivity gives

$$\begin{aligned} H(M_1, M_2) &= H(Z) + H(M_1, M_2 \mid Z) \\ &\le H(Z) + H(M_1 \mid Z) + H(M_2 \mid Z) \\ &= H(M_1) + H(M_2) - H(Z). \end{aligned}$$

Conversely, take entropy-maximizing distributions on the two contractions. Their copies of Z have the common law $(x_e)_{e \in \partial X}$. Coupling them independently conditional on Z produces compatible pairs, hence perfect matchings of G, with all prescribed edge marginals. This coupling attains equality. Hence

$$H_{\max}(G, x) = H_{\max}(G_1, x^{(1)}) + H_{\max}(G_2, x^{(2)}) - H(Z). \tag{4.4}$$

Every noncut edge occurs in exactly one contraction, whereas every cut edge occurs in both. Therefore

$$S_G(x) = S_{G_1}(x^{(1)}) + S_{G_2}(x^{(2)}) - H(Z). \tag{4.5}$$

Subtracting (4.4) from (4.5) proves (4.3). □

The removable-set criterion across a tight cut [20, Propositions 8.7 and 8.8], applied to singleton edge sets, yields the following lifting rule.

Lemma 4.2. *Let G_1, G_2 be the contractions of a tight cut C in an elementary labeled bipartite multigraph G.*

(i) If $e \notin C$ is removable in the contraction containing it, then e is removable in G.
(ii) If $e \in C$ is removable in both contractions, then e is removable in G.

Proof. We give the short splicing argument. In case (i), the two contractions of $G - e$ are $G_1 - e$ and G_2, after interchanging the indices if necessary; in case (ii), they are $G_1 - e$ and $G_2 - e$. Under the stated hypotheses both contractions are elementary. Given any edge label of $G - e$, choose a perfect matching containing its copy in the appropriate contraction. The cut-edge label used by that matching is admissible in the other contraction, so one may choose a matching there with the same cut-edge label and splice the two matchings. Thus every edge of $G - e$ is admissible. Connectivity follows from the connectivity of the two contractions and the presence of their common cut-edge labels. Hence $G - e$ is elementary in either case. □

This lifting rule makes the removable-edge correction compatible with tight-cut induction.

4.2. **Tight-cut contractions to braces.** An elementary bipartite multigraph with no nontrivial tight cut is called a *brace* [5, Section 1.1.2]. The following standard minimal-shore observation supplies the leaf used in the induction; compare [5, proof of Corollary 16].

Lemma 4.3. *If an elementary bipartite multigraph G is not a brace, then G has a nontrivial tight shore X such that contracting $V(G) \setminus X$ yields a brace.*

Proof. Choose an inclusion-minimal nontrivial tight shore X, and let z be the vertex obtained by contracting its complement. If the resulting contraction had a nontrivial tight cut, choose the shore of that cut not containing z. The standard correspondence of tight cuts under contraction would lift it to a nontrivial tight shore of G properly contained in X, a contradiction. Thus the contraction has no nontrivial tight cut and is a brace. □

We refer to the contraction in Lemma 4.3 as a brace-producing contraction.

Carvalho, Lucchesi, and Murty proved that every edge of a brace with at least six vertices is removable; see Corollary 3.9 of [4] and Theorem 15 of [5]. We use the result in the following labeled form.

Lemma 4.4. *Let G be a labeled bipartite brace of bipartite order $k \geq 3$. Then every labeled edge of G is removable.*

Up to isomorphism, C_4 is the unique simple brace on four vertices [5, Section 1.1.2]. Its exceptional behavior points to the correct potential formula: separate a perfect matching into its choice of one of the two opposite-edge patterns and its subsequent choices of labels inside the selected bundles. The Shannon chain rule [7, Sections 2.2 and 2.5] then makes the conditional label entropies cancel from $S_G - H_{\max}$, leaving only the entropy of the binary pattern choice.

Lemma 4.5. *Let G be an elementary labeled bipartite multigraph whose underlying simple graph is a 4-cycle, and let x be a strictly positive fractional perfect matching of G. Choose a pair of opposite edge bundles, and let $t \in (0,1)$ be the common total x-weight of the two bundles; equivalently, t is the probability of the corresponding perfect-matching pattern. Then*

$$\Phi(G,x) = h(t) \le \log 2. \tag{4.6}$$

Proof. Label the four bundles cyclically by B_1, B_2, B_3, B_4, with $\{B_1, B_3\}$ the chosen opposite pair. Their total weights alternate as $t, 1-t, t, 1-t$. Let π_ℓ be the conditional distribution of the labels inside B_ℓ. A perfect matching chooses the pattern $\{B_1, B_3\}$ with probability t and the pattern $\{B_2, B_4\}$ with probability $1-t$. Conditional on the pattern, entropy is maximized by choosing the two labels independently. Hence

$$\begin{aligned} H_{\max}(G,x) = h(t) &+ t\big(H(\pi_1) + H(\pi_3)\big) \\ &+ (1-t)\big(H(\pi_2) + H(\pi_4)\big), \end{aligned}$$

whereas the two local row entropies give

$$\begin{aligned} S_G(x) = 2h(t) &+ t\big(H(\pi_1) + H(\pi_3)\big) \\ &+ (1-t)\big(H(\pi_2) + H(\pi_4)\big). \end{aligned}$$

Subtracting yields $\Phi(G,x) = h(t) \le \log 2$. □

The quadrilateral case cannot use the rooted brace estimate because its cut-edge bundles need not be removable. We use a bundle version of bicontraction at the vertex w, which has two distinct neighbors: delete w, merge those neighbors, and retain all external edge labels. If each of the two bundles at w is a singleton, this is the standard bicontraction of [5, Section 1.1.3]; parallel labels require the explicitly defined bundle version. The operation lowers the bipartite order by one. In a perfect matching of the contracted graph, the edge incident with the identified vertex has a binary origin variable indicating whether it came from the u- or v-side; the entropy chain rule assigns this variable the term $h(\alpha)$. At most one removable edge label of the contracted graph can fail to lift, and its weight is α or $1-\alpha$, so its contribution is bounded by $\delta(\alpha)$. These complementary terms are exactly what is needed for the induction.

Lemma 4.6. *Let G be a nonbrace elementary labeled bipartite multigraph of bipartite order k, and let x be a strictly positive fractional perfect matching of G. Suppose that a tight-cut contraction of G has underlying simple graph C_4. Then there exist an elementary labeled bipartite multigraph G' of bipartite order $k-1$, a strictly positive fractional perfect matching x' of G', and a number $\alpha \in (0,1)$ such that*

$$\Phi(G,x) = \Phi(G',x') + h(\alpha) \tag{4.7}$$

and

$$\sum_{e \in \mathcal{R}(G')} \delta(x'_e) \le \sum_{e \in \mathcal{R}(G)} \delta(x_e) + \delta(\alpha). \tag{4.8}$$

Proof. Step 1: construction. Let z be the contraction vertex of the C_4 factor, let w be the vertex opposite z, and let u, v be the two remaining vertices. Up to interchanging the color classes, $w \in L$ and $u, v \in R$. In the original graph, every

edge at w joins it to either u or v. Let D_u, D_v be these two labeled edge bundles and set

$$\alpha = \sum_{e \in D_u} x_e, \qquad 1 - \alpha = \sum_{e \in D_v} x_e. \tag{4.9}$$

Both bundles are nonempty and all their edge weights are positive, so $\alpha \in (0, 1)$.

Delete w, merge u and v into a single vertex q, and retain every edge outside $D_u \cup D_v$, together with its label and weight. The resulting graph G' has bipartite order $k - 1$. Denote the retained weights by x'. By the matching equations and (4.9), the total weight of the external edges formerly incident with u is $1 - \alpha$, while the total weight of those formerly incident with v is α. Thus x' is strictly positive and satisfies the fractional perfect-matching equations, including the equation at q.

Every perfect matching of G projects to a perfect matching of G'. Hence every retained edge lies in a perfect matching of G', because it lies in a perfect matching of the elementary graph G. Conversely, if the edge at q in a matching of G' originated at v, one completes the matching in G by choosing a label from D_u; if it originated at u, one chooses a label from D_v. Finally, paths in G project to walks in G', with every occurrence of the segment u, w, v collapsed to q; hence G' is connected. Thus G' is elementary.

Step 2: entropy identities. Write $\pi^u = (x_e/\alpha)_{e \in D_u}$ and $\pi^v = (x_e/(1 - \alpha))_{e \in D_v}$. For a random perfect matching M of G with marginals x, let M' be its projection to G', let $O \in \{u, v\}$ record the origin of the edge at q, and let B be the label of the edge of M incident with w. Because the original edge labels are retained, O is determined by M'. Moreover, $\Pr(O = v) = \alpha$; conditional on $O = v$, the label B has distribution π^u, and conditional on $O = u$, it has distribution π^v. Since (M', B) determines M,

$$\begin{aligned} H(M) &= H(M') + H(B \mid M') \\ &\le H(M') + H(B \mid O) \\ &\le H_{\max}(G', x') + \alpha H(\pi^u) + (1 - \alpha) H(\pi^v). \end{aligned}$$

Conversely, start with an entropy-maximizing distribution in $\mathcal{P}(G', x')$, and, conditional on O, choose B independently according to π^u or π^v. The lifted distribution has every prescribed edge marginal and attains the upper bound. Therefore

$$H_{\max}(G, x) = H_{\max}(G', x') + \alpha H(\pi^u) + (1 - \alpha) H(\pi^v). \tag{4.10}$$

Summing local entropies over the left color class gives

$$S_G(x) = S_{G'}(x') + h(\alpha) + \alpha H(\pi^u) + (1 - \alpha) H(\pi^v). \tag{4.11}$$

Subtracting (4.10) from (4.11) proves (4.7).

Step 3: removable edges. It remains to compare removable edges. Let E_u and E_v be the external labeled edges at u and v, respectively, so that their total weights are $1 - \alpha$ and α. Suppose that $f \in \mathcal{R}(G')$. If both $E_u \setminus \{f\}$ and $E_v \setminus \{f\}$ are nonempty, then $f \in \mathcal{R}(G)$. Indeed, every retained edge of $G - f$ lies in a perfect matching obtained by lifting a perfect matching of $G' - f$. To obtain a perfect matching containing a prescribed label in D_u, choose a label in $E_v \setminus \{f\}$. Since $G' - f$ is elementary, that label lies in a perfect matching of $G' - f$, which may then be lifted using the prescribed label in D_u. The symmetric argument, using a label in $E_u \setminus \{f\}$, permits any prescribed label in D_v. Thus every edge of $G - f$

is admissible. Since both external groups remain nonempty, expanding q into the path u, w, v also preserves connectivity. Hence $G - f$ is elementary.

Consequently, an edge removable in G' but not in G must be the unique labeled edge in E_u or the unique labeled edge in E_v. There cannot be one exceptional removable edge in each group. If both groups were singletons, q would have labeled degree two in G'; deleting either incident edge would leave a vertex of degree one. Because G is not a brace, $k \geq 3$, and hence G' has bipartite order at least two. As observed above, an elementary graph of such order has minimum labeled degree at least two. Thus neither singleton edge would be removable in G', and there is at most one exceptional edge. Its weight is either α or $1 - \alpha$, and $\delta(\alpha) = \delta(1 - \alpha)$.

For completeness, set $Q = \mathcal{R}(G') \backslash \mathcal{R}(G)$, where both removable sets are restricted to the retained edge labels. We have proved that $|Q| \leq 1$, that $x'_e = x_e$ for every retained label, and that, if $Q = \{f\}$, then $x'_f \in \{\alpha, 1 - \alpha\}$. Since $\delta \geq 0$, it follows that

$$\sum_{e \in \mathcal{R}(G')} \delta(x'_e) \leq \sum_{e \in \mathcal{R}(G)} \delta(x_e) + \delta(\alpha),$$

which is (4.8). □

4.3. **The potential theorem.** The local estimates now determine the induction. A brace factor whose underlying simple graph is not C_4 is handled by rooting the capacity estimate at its contraction vertex, and tight-cut additivity joins it to the remaining contraction without loss. A C_4 factor is handled by the bicontraction described above, with parallel labels retained: lowering the bipartite order releases one copy of $\log 2$, exactly balancing

$$h(\alpha) + \delta(\alpha) = \log 2.$$

This gives the global inequality.

Theorem 4.7. *Let G be an elementary labeled bipartite multigraph of bipartite order k, and let x be a strictly positive fractional perfect matching of G. Then*

$$\Phi(G, x) \leq (k - 1) \log 2 + \sum_{e \in \mathcal{R}(G)} \delta(x_e). \tag{4.12}$$

Proof. We argue by induction on k. If $k = 1$, a perfect matching consists of a single labeled edge from the unique parallel class. Its maximum entropy is exactly $S_G(x)$, so $\Phi(G, x) = 0$.

Step 1: braces. Suppose $k \geq 2$. If G is a brace whose underlying simple graph is C_4, Lemma 4.5, in the form (4.6), gives

$$\Phi(G, x) \leq \log 2 = (k - 1) \log 2.$$

If G is any other brace, then $k \geq 3$; fix a right vertex z. Lemma 4.4 says that every edge is removable. Proposition 3.7 therefore yields

$$\Phi(G, x) \leq (k - 1) \log 2 + \sum_{\substack{e \in \mathcal{R}(G) \\ e \not\ni z}} \delta(x_e),$$

which is stronger than (4.12).

Step 2: a nonbrace graph with a nonquadrilateral brace factor. It remains to consider a nonbrace graph. By Lemma 4.3, choose a brace-producing contraction G_1, let G_2 be the other tight-cut contraction, and let their bipartite orders be k_1, k_2. The inequalities $2 \leq k_i \leq k - 1$ established above make induction applicable

to both contractions. First suppose that the underlying simple graph of G_1 is not C_4. When applying Proposition 3.7, interchange the two color classes if necessary and take the contraction vertex as the right root. The rooted estimate assigns no correction to cut edges. Using the exact additivity (4.3) from Lemma 4.1, together with the induction hypothesis for G_2, we obtain

$$\begin{aligned}\Phi(G,x) &\le (k_1-1)\log 2 + (k_2-1)\log 2 \\ &\quad + \sum_{\substack{e\in E(G_1)\\ e \text{ is not a cut edge}}} \delta(x_e) + \sum_{e\in\mathcal{R}(G_2)} \delta(x_e).\end{aligned} \tag{4.13}$$

Every noncut edge occurring on the G_1-side is removable in G_1, hence lifts to a removable edge of G. A noncut edge in $\mathcal{R}(G_2)$ also lifts. If an edge of $\mathcal{R}(G_2)$ is a cut edge, it is removable in G_1 by Lemma 4.4, and therefore lifts by Lemma 4.2. The two sums in (4.13) involve disjoint edge labels: the first consists of noncut edges on the G_1-side, while the second consists of edges on the G_2-side, including any cut edge. Thus their sum is bounded by the full sum over $\mathcal{R}(G)$. Using (4.1), we obtain (4.12).

Step 3: a quadrilateral brace factor. Finally, suppose that the brace factor has underlying simple graph C_4. Apply Lemma 4.6 and the induction hypothesis to the graph G' of bipartite order $k-1$:

$$\begin{aligned}\Phi(G,x) &= \Phi(G',x') + h(\alpha) \\ &\le (k-2)\log 2 + \sum_{e\in\mathcal{R}(G')} \delta(x'_e) + h(\alpha) \\ &\le (k-2)\log 2 + \sum_{e\in\mathcal{R}(G)} \delta(x_e) + \delta(\alpha) + h(\alpha) \\ &= (k-1)\log 2 + \sum_{e\in\mathcal{R}(G)} \delta(x_e).\end{aligned}$$

The last equality follows from $\delta(\alpha) + h(\alpha) = \log 2$ and (3.8). This completes the induction. □

The correction sum in Theorem 4.7 is empty when G is minimal elementary. That is the decisive point for nearly decomposable matrices.

5. Proof of the sharp permanent bound

We now return to the original matrix problem. The deduction from the potential theorem is short, but it is useful to display it separately to make clear where the nearly decomposable hypothesis is used.

We now prove Theorem 1.2.

Proof. Let $G = G(A)$, and assign the edge-weight vector $x = (a_{ij})_{ij\in E(G)}$. By Corollary 2.2, G is elementary and has no removable edge. Hence Theorem 4.7 gives

$$\Phi(G,x) \le (n-1)\log 2.$$

Lemma 2.4, in the form (2.8), now yields

$$\operatorname{per} A \ge \exp\bigl(-\Phi(G,x)\bigr) \ge \exp\bigl(-(n-1)\log 2\bigr) = 2^{1-n}.$$

To prove sharpness, let P be the permutation matrix corresponding to a cycle of length n, and let I denote the identity matrix. Set

$$A = \frac{1}{2}(I + P).$$

The support graph of A is a cycle of length $2n$. It is elementary, while deleting any support edge destroys the matching-covered property. Thus A is nearly decomposable. The only two nonzero terms in its permanent expansion are those corresponding to I and P, and each equals 2^{-n}. Therefore

$$\operatorname{per} A = 2 \cdot 2^{-n} = 2^{1-n}.$$

□

The proof also identifies a mechanism beyond the stated conjecture: the contribution assigned to each removable edge is quantified by its binary entropy deficit. This strengthened form is what makes the estimate stable under tight-cut contraction, even though the class of minimal elementary supports is not preserved by the local reductions used in the proof.

Declarations

The author declares that there is no conflict of interest. No data were generated or analyzed in this study.

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