

GAUSSIAN DECAY AND BACKWARD UNIQUENESS ON DOMAINS WITH CONNECTED BOUNDARY

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ABSTRACT. For every $n \geq 3$ we construct an unbounded domain in $\mathbb{R}^n$ with smooth connected boundary for which the attainable Gaussian decay rates of nonzero harmonic functions form a nonempty bounded interval. The proposed equivalence between Gaussian harmonic decay and failure of backward uniqueness for bounded temperatures is therefore false, even with this boundary condition. The argument identifies two dilates of one domain, at least one of which violates the equivalence. In the plane, harmonic decay of order $e^{-a|z|^q}$ on an unbounded Jordan domain implies that its Riemann maps belong to every H^p with $p < q$. On unbounded quasidisks, the exact criterion is H^q membership, and every positive decay rate is then attainable.

1. INTRODUCTION

Backward uniqueness on a domain without a prescribed lateral boundary condition is sensitive to its geometry at infinity. The result is not a consequence of the usual uniqueness theorem for a fixed initial or boundary value problem. For exterior domains, this distinction appears in the work of Escauriaza, Seregin, and Šverák [ESS03]. For cones, Li and Šverák [LS12, Section 1, Theorem 1.1] record Escauriaza's nonuniqueness construction for opening angles below 90° and prove uniqueness above $2\arccos(1/\sqrt{3})$. Wu and Wang [WW15, Theorem 1.1] and Rüländ [Rul15, Proposition 1.2] subsequently improved the angle range. These statements impose no lateral boundary data.

Decay of harmonic functions has its own geometric uniqueness theory. For example, Arshon [Ars68] studied three-dimensional solids of revolution, obtaining analogues of analytic-function decay theorems. The problem considered here asks whether a single harmonic decay condition also characterizes the heat equation's failure of backward uniqueness. Eremenko [Ere15, p. 1] formulates the question below and attributes the equivalence to a 1983 announcement of Gurarii and Matsaev [GM83]. Li and Šverák [LS12, Section 1] note that the announced conical classification has no proof in print. Rüländ [Rul15, Remark 4] makes the same observation about the heat-to-Laplace reduction. Our point of departure is the explicit formulation in Eremenko's note, rather than an interpretation of unstated hypotheses in the earlier announcement.

Let $D \subset \mathbb{R}^n$ be a domain. A *bounded temperature* on $D \times [0, 1]$ is a bounded real function $u \in C^{2,1}(D \times (0, 1)) \cap C(\overline{D \times [0, 1]})$ satisfying $u_t = \Delta u$. We use the following notation:

- (1) $I(D) : \text{some such } u \not\equiv 0 \text{ satisfies } u(\cdot, 1) = 0;$
 $II(D) : \text{some } h \in \mathcal{H}(D) \setminus \{0\} \text{ satisfies } h(X) = O(e^{-|X|^2}) \text{ as } X \rightarrow \infty \text{ in } D.$

Here $\mathcal{H}(D)$ denotes the real harmonic functions on D . The second estimate means that $|h(X)| \leq Ce^{-|X|^2}$ whenever $X \in D$ and $|X| > R$, for some $C, R > 0$. No bound near a finite boundary point is included in this condition.

Problem 1. Are the two properties in (1) equivalent on every domain whose finite boundary points are regular for the Dirichlet problem for the Laplacian? Does the equivalence hold if the boundary is required to be smooth and connected?

We give a negative answer to both questions in every dimension $n \geq 3$. All boundary regularity in this paper concerns finite points of $\partial D \subset \mathbb{R}^n$; no additional condition is imposed at a compactified point at infinity.

For an unbounded domain D , define its set of attainable Gaussian rates and its supremum by

- (2) $\mathcal{E}(D) = \{a > 0 : \exists h \in \mathcal{H}(D) \setminus \{0\}, h(X) = O(e^{-a|X|^2}) \text{ in } D\},$
 $\gamma(D) = \sup(\{0\} \cup \mathcal{E}(D)).$

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The set $\mathcal{E}(D)$ is downward closed. Its upper endpoint need not be attained. Put $a_* = (\log 2)/64$.

Theorem 2. *For every integer $n \geq 3$ there is an unbounded domain $D \subset \mathbb{R}^n$ with C^∞ connected boundary such that*

$$(0, a_*] \subset \mathcal{E}(D) \subset (0, 20].$$

There is a nonzero $H \in \mathcal{H}(D)$ with $|H(X)| \leq Ce^{-a_*|X|^2}$ for every $X \in D$.

The constants are deliberately not optimized. The distinction that matters is between a finite positive value of γ and the values zero and infinity. The explicit attainment of a_* fixes a convenient smaller dilate in the next statement.

Corollary 3. *For every $n \geq 3$ there are two unbounded domains $D_-, D_+ \subset \mathbb{R}^n$ with smooth connected boundaries such that*

$$\Pi(D_-), \quad -\Pi(D_+), \quad \text{I}(D_-) \implies \text{I}(D_+).$$

At least one of the domains in Corollary 3 therefore violates the equivalence in Problem 1.

The proof takes $D_- = \sqrt{a_*} D$ and $D_+ = \sqrt{40} D$ for the domain in Theorem 2. It does not determine the truth value of I on either domain. In particular, the corollary neither supplies a terminally vanishing temperature on a specified member of the pair nor asserts that both implications fail separately. This logical distinction does not affect the negative answer to the universal equivalence.

The construction has two analytic parts. Signed measures on codimension-two spheres cancel progressively more harmonic moments; their Newtonian potential gives the lower bound. The upper bound applies to every harmonic function, not only to the rotationally invariant potential used for existence. It follows by projecting onto all transverse spherical harmonics, reducing each coefficient to a first-order complex equation, and applying a maximum principle on a perforated half-plane. Finally, removability of curves and compactness provide a neighborhood within which the obstacles may be joined. No continuity of γ under arbitrary boundary perturbations is assumed.

The planar results are different. Write $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, $\mathbb{H} = \{w \in \mathbb{C} : \text{Im } w > 0\}$, and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. An *unbounded Jordan domain* is a component of $\widehat{\mathbb{C}} \setminus J$, viewed as a plane domain, where J is a Jordan curve containing ∞ . For a holomorphic function f on $\mathbb{D}$, write

$$m_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \quad M(r, f) = \max_{|\zeta| \leq r} |f(\zeta)|.$$

Membership in H^p means that $\sup_{r < 1} m_p(r, f) < \infty$. The relation between Hardy classes and the geometry of unbounded quasidisks is studied by Kim and Sugawa [KS11, Lemma 2.1 and Sections 2–4, pp. 292–298]. Our hypothesis concerns the decay of a real harmonic function, not the modulus of a given holomorphic function.

Theorem 4. *Let D be an unbounded Jordan domain and let $q > 0$. If some $h \in \mathcal{H}(D) \setminus \{0\}$ satisfies*

$$(3) \quad h(z) = O(e^{-a|z|^q}) \quad (z \rightarrow \infty, z \in D)$$

for some $a > 0$, every conformal map $\phi : \mathbb{D} \rightarrow D$ belongs to H^p for every $0 < p < q$.

For the endpoint, an *unbounded quasidisk* means a domain admitting a conformal map $F : \mathbb{H} \rightarrow D$ with a quasiconformal extension to $\widehat{\mathbb{C}}$ fixing infinity. This is the standard unbounded version of the quasidisk condition; see Kim and Sugawa [KS11, pp. 291–292, 296–298].

Theorem 5. *Let D be an unbounded quasidisk, $\phi : \mathbb{D} \rightarrow D$ conformal, and $q > 0$. The following conditions are equivalent.*

- (i) *Some $h \in \mathcal{H}(D) \setminus \{0\}$ satisfies (3) for some $a > 0$.*
- (ii) *$\phi \in H^q$.*
- (iii) *For every $A > 0$ there is $h_A \in \mathcal{H}(D) \setminus \{0\}$ with $|h_A(z)| \leq e^{-A|z|^q}$ for every $z \in D$.*

Let $\mathfrak{h}(D) = \sup\{p > 0 : \phi \in H^p\}$ be the Hardy number of a proper simply connected domain. The harmonic-majorant characterization below also shows that this number is independent of the Riemann map.

Corollary 6. *For an unbounded quasidisk D , one has $\gamma(D) = \infty$ if its Riemann maps belong to H^2 , and $\gamma(D) = 0$ otherwise.*

Outside that class there is still an endpoint restriction.

Corollary 7. *If an unbounded Jordan domain D satisfies $0 < \gamma(D) < \infty$, then $\mathfrak{h}(D) = 2$, its Riemann maps do not belong to H^2 , and D is not an unbounded quasidisk.*

These planar statements do not decide backward uniqueness for the heat equation on a planar domain with smooth connected boundary. That question is isolated at the end of the paper.

Organization. Section 2 gives the scaling argument and basic potential theory. Sections 3–4 construct the rapidly decaying potential. Section 5 proves the upper bound, and Section 6 joins the obstacles and completes Theorem 2 and Corollary 3. Section 7 collects the planar analytic facts. Sections 8 and 9 prove the two planar theorems. Section 10 discusses their consequences and the remaining planar problem.

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This paper has not yet been carefully polished, and there are currently no plans to submit it for publication.

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2. SCALING AND BASIC POTENTIAL THEORY

Only one direction of the scaling law for temperatures is needed. The following proof remains inside the original time interval and requires no extension across a terminal time.

Lemma 8. *For every domain $G \subset \mathbb{R}^n$ and every $\lambda \geq 1$, $I(G)$ implies $I(\lambda G)$.*

Proof. Let u witness $I(G)$, and put $b = \sup\{t \in [0, 1] : u(\cdot, t) \not\equiv 0\}$. Continuity gives $b > 0$ and $u(\cdot, b) = 0$; the latter also follows from the assumed terminal condition if $b = 1$. Set $s = \max\{0, b - \lambda^{-2}\}$. Then $s + \lambda^{-2} \leq 1$, $u(\cdot, s + \lambda^{-2}) = 0$, and u is nonzero somewhere in this closed time interval. Define

$$(4) \quad U(X, t) = u(X/\lambda, s + t/\lambda^2), \quad X \in \lambda G, \quad 0 \leq t \leq 1.$$

Equation (4) gives $U_t = \lambda^{-2}u_t = \Delta_X U$. Boundedness, continuity at the time endpoints, and the required interior differentiability follow from those of u . Thus U witnesses $I(\lambda G)$. $\square$

Harmonic functions obey the exact rate identity

$$(5) \quad \mathcal{E}(\lambda G) = \lambda^{-2}\mathcal{E}(G), \quad \gamma(\lambda G) = \lambda^{-2}\gamma(G) \quad (\lambda > 0).$$

Indeed, composition with $X \mapsto X/\lambda$ and its inverse preserves harmonicity and rescales the exponent as in (5). Lemma 8 explains why a finite positive harmonic threshold obstructs the universal heat–harmonic equivalence.

We write $B(X, r)$ for a Euclidean ball, $\mathbb{S}^{d-1}$ for the unit sphere in $\mathbb{R}^d$, and dA for planar Lebesgue measure. Surface measure is normalized to total mass one unless its total area is explicitly written as $|\mathbb{S}^{d-1}|$. The symbol $\delta_{\ell m}$ is one when $\ell = m$ and zero otherwise. Constants denoted by C_n may depend on the fixed dimension; their values may change at different occurrences. Constants never depend on a running shell index or evaluation point unless indicated. We need three classical facts in the following forms.

Lemma 9 ([ABR01, Theorems 2.4 and 2.6, pp. 33–36]). *For a harmonic function v on $B(X, r) \subset \mathbb{R}^n$,*

$$(6) \quad |\nabla v(X)| \leq \frac{C_n}{r} \sup_{B(X, r)} |v|.$$

A locally uniformly bounded sequence of harmonic functions on a fixed open set has a locally uniformly convergent subsequence, with harmonic limit.

The local form of curve removability follows by applying the cited compact-set theorem in successively smaller balls.

Lemma 10 ([ABR01, Theorem 9.13, pp. 202–203]). *Let $n \geq 3$. A relatively closed set that is locally a finite union of C^1 compact arcs is removable for locally bounded harmonic functions.*

At an intersection of finitely many arcs, remove them successively; the bounded extensions agree off the arcs and hence agree everywhere. The hypothesis $n \geq 3$ is essential to this use of Lemma 10.

Lemma 11 ([ABR01, Theorem 5.12, p. 81]). *For $d \geq 2$, a spherical harmonic of degree ℓ is the restriction to $\mathbb{S}^{d-1}$ of a homogeneous harmonic polynomial of degree ℓ . The spaces of spherical harmonics of distinct degrees are mutually orthogonal, and their algebraic sum is dense in $L^2(\mathbb{S}^{d-1})$ with normalized surface measure.*

3. ROTATIONAL OBSTACLES AND QUADRATURE

Fix $n \geq 3$ for Sections 3–6. Write $X = (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $r = |y|$, and $z = x + ir$. Choose R_1 to be a power of two with $R_1 \geq \max\{2, \sqrt{n}\}$. Set

$$(7) \quad R_k = 2^{k-1} R_1, \quad L_k = R_k^2, \quad N_k = 2L_k + n - 1, \quad \varepsilon_k = e^{-R_k^6}.$$

In particular $N_k + 1 \leq 3R_k^2$. Define

$$(8) \quad \theta_{k,j} = \begin{cases} (j - \frac{1}{2})\pi/N_k, & n \text{ odd}, \\ j\pi/(N_k + 1), & n \text{ even}, \end{cases} \quad \lambda_{k,j} = R_k e^{i\theta_{k,j}} = \xi_{k,j} + i\eta_{k,j}, \quad 1 \leq j \leq N_k.$$

We denote the set of these upper half-plane nodes by Λ and write $\Lambda^\# = \{\bar{\lambda} : \lambda \in \Lambda\}$ for its complex conjugate. The lower bounds

$$(9) \quad \eta_{k,j} \geq \frac{1}{3R_k}, \quad |\lambda_{k,j} - \lambda_{k,j+1}| \geq \frac{2}{3R_k}$$

follow from $\sin t \geq 2t/\pi$ on $[0, \pi/2]$ and (7)–(8).

Let $\Sigma_{k,j} = \{(\xi_{k,j}, \eta_{k,j}\omega) : \omega \in \mathbb{S}^{n-2}\}$. The basic obstacles and their exterior are

$$(10) \quad \begin{aligned} K_{k,j} &= \{X : \text{dist}(X, \Sigma_{k,j}) \leq \varepsilon_k\}, \\ K_b &= \overline{B(0, 1/2)} \cup \bigcup_{k,j} K_{k,j}, \quad \Omega = \mathbb{R}^n \setminus K_b. \end{aligned}$$

The sets $K_{k,j}$ are pairwise disjoint smooth compact domains, diffeomorphic to $\mathbb{S}^{n-2} \times \overline{\mathbb{D}}$, where the last factor is a two-dimensional closed disk. Their boundaries are connected. The family is locally finite by (7). The distance to a central sphere is exactly

$$(11) \quad \text{dist}((x, r\omega), \Sigma_{k,j}) = |x + ir - \lambda_{k,j}|.$$

The domain Ω is connected. To see this, start with a compact polygonal path and detour around the finitely many obstacles it meets, using a small exterior collar of each connected boundary. The collars can be chosen disjoint. The same argument proves

$$(12) \quad \Omega \cap \{|X| > s_m\} \text{ is connected,} \quad s_m = \frac{3}{2} R_m.$$

Indeed, each obstacle lies entirely on one side of the sphere $|X| = s_m$, and exterior collars of the obstacles outside that sphere can be chosen to remain outside it.

For the moment construction we use zonal harmonics. Put $\nu = (n-2)/2$ and

$$Q_\ell(t) = \frac{C_\ell^{(\nu)}(t)}{C_\ell^{(\nu)}(1)}, \quad d\omega_n(t) = c_n(1-t^2)^{(n-3)/2} dt, \quad c_n = \frac{\Gamma(n/2)}{\sqrt{\pi} \Gamma((n-1)/2)}.$$

Here Γ is the gamma function, and $C_\ell^{(\nu)}$ is the Gegenbauer polynomial. The next formulas fix all normalizations. They follow from the standard orthogonality and generating formulas in the NIST reference and the zonal reproducing kernel.

Lemma 12 ([NIST, Table 18.3.1, (18.12.4)]; [ABR01, Proposition 5.27, pp. 95–96]). *Let $Z_\ell(X) = |X|^\ell Q_\ell(x/|X|)$, extended as a polynomial at zero, and put $d_{\ell,n} = (2\ell + n - 2)(\ell + n - 3)!/[\ell!(n-2)!]$. Then Z_ℓ is harmonic, $|Q_\ell| \leq 1$ on $[-1, 1]$, and*

$$(13) \quad \int_{-1}^1 Q_\ell(t) Q_m(t) d\omega_n(t) = \frac{\delta_{\ell m}}{d_{\ell,n}}.$$

For $R > 0$ and $-1 < t < 1$, let $\sigma_{R,t}$ be probability surface measure on $\{(Rt, R\sqrt{1-t^2}\omega) : \omega \in \mathbb{S}^{n-2}\}$. For $|X| > R$,

$$(14) \quad \int |X - Y|^{2-n} d\sigma_{R,t}(Y) = \sum_{\ell=0}^{\infty} \frac{C_\ell^{(\nu)}(1) R^\ell}{|X|^{\ell+n-2}} Q_\ell(x/|X|) Q_\ell(t).$$

The series converges locally uniformly for $|X| > R$.

One way to verify the normalization in (14) is to expand the Newtonian kernel with the Gegenbauer generating function. Averaging its degree- ℓ term over rotations fixing the x axis gives an invariant spherical

harmonic in the X variable, hence a multiple of $Q_\ell(x/|X|)$. Evaluation at the positive axis gives the multiplier $C_\ell^{(\nu)}(1)Q_\ell(t)$. We will use the polynomial bounds

$$(15) \quad d_{\ell,n} \leq C_n(1+\ell)^{n-2}, \quad C_\ell^{(\nu)}(1) = \frac{(\ell+n-3)!}{\ell!(n-3)!} \leq C_n(1+\ell)^{n-3}.$$

The quadrature required below is classical. Fejér [Fej33] studied quadrature with positive weights. The explicit cosine-midpoint weights used here are given by Waldvogel [Wal06, Section 2, (2.3), pp. 196–197]. The other case is the second-kind Gauss–Chebyshev rule; see the NIST reference [NIST, Section 3.5(v)].

Lemma 13. *For the nodes $t_{k,j} = \cos \theta_{k,j}$ of (8), there are positive numbers $w_{k,j}$ with $\sum_j w_{k,j} = 1$ such that*

$$(16) \quad \sum_j w_{k,j} q(t_{k,j}) = \int_{-1}^1 q(t) d\omega_n(t) \quad (\deg q \leq 2L_k).$$

Proof. For odd n , let $N = N_k$ and $\theta_j = (j - 1/2)\pi/N$. The weights

$$v_j = \frac{1}{N} \left(1 - 2 \sum_{h=1}^{\lfloor (N-1)/2 \rfloor} \frac{\cos(2h\theta_j)}{4h^2 - 1} \right)$$

integrate every polynomial of degree at most $N - 1$ against $dt/2$. Indeed, discrete cosine orthogonality and $(1/2) \int_{-1}^1 \cos(2h \arccos t) dt = -1/(4h^2 - 1)$ give the formula on a polynomial basis. Moreover, $2 \sum_{h=1}^m (4h^2 - 1)^{-1} = 1 - (2m + 1)^{-1} < 1$, so $v_j > 0$. Take $w_{k,j} = 2c_n v_j (1 - t_{k,j}^2)^{(n-3)/2}$. For $\deg q \leq 2L_k$, the product integrated has degree at most $2L_k + n - 3 \leq N_k - 1$, proving (16).

For even n , the nodes are $t_j = \cos(j\pi/(N+1))$. The positive weights $\pi \sin^2(j\pi/(N+1))/(N+1)$ integrate polynomials of degree at most $2N - 1$ against $\sqrt{1-t^2} dt$. This follows by discrete sine orthogonality and division by the second-kind Chebyshev polynomial U_N , defined by $U_N(\cos \theta) = \sin((N+1)\theta)/\sin \theta$ and orthogonal to all polynomials of degree below N . Multiply these weights by $c_n(1-t_j^2)^{(n-4)/2}$. The additional factor is a polynomial of degree $n - 4$, so exactness includes every degree at most $2L_k$. This proves (16) in the even case as well. $\square$

4. A GAUSSIAN-DECAYING NEWTONIAN POTENTIAL

Let $\sigma_{k,j}$ be probability surface measure on the sphere $\Sigma_{k,j}$ defined in (10). For a finite signed measure μ , $\|\mu\|_{\text{TV}}$ denotes its total variation. All measures in this section are invariant under rotations of the transverse variable y .

Proposition 14. *There is a nonzero $H \in \mathcal{H}(\Omega)$ such that*

$$(17) \quad |H(X)| \leq C_n e^{-a_*|X|^2} \quad (X \in \Omega).$$

Proof. Step 1. Exact cancellation. Start with $\mu_0 = \delta_0$, the unit point mass at zero. Use the polynomials in Lemma 12. Given μ_{k-1} , define $m_\ell^{(k-1)} = \int Z_\ell d\mu_{k-1}$ and

$$(18) \quad \begin{aligned} f_k(t) &= - \sum_{\ell=0}^{L_k} d_{\ell,n} \frac{m_\ell^{(k-1)}}{R_k^\ell} Q_\ell(t), \\ \nu_k &= \sum_{j=1}^{N_k} w_{k,j} f_k(t_{k,j}) \sigma_{k,j}, \quad \mu_k = \mu_{k-1} + \nu_k. \end{aligned}$$

The product $f_k Q_m$ has degree at most $2L_k$ if $m \leq L_k$. Equations (13) and (16) therefore imply $\int Z_m d\nu_k = -m_m^{(k-1)}$, and hence

$$(19) \quad \int Z_m d\mu_k = 0 \quad (0 \leq m \leq L_k).$$

This is an exact identity, not a quadrature error estimate.

Step 2. Summable masses. We have $f_1 = -1$ and $\|\nu_1\|_{\text{TV}} = 1$. For $k \geq 2$, (19) removes all indices $\ell \leq L_{k-1}$ from (18). Since $\text{supp } \mu_{k-1} \subset \overline{B(0, R_{k-1})}$, $|m_\ell^{(k-1)}| \leq \|\mu_{k-1}\|_{\text{TV}} R_{k-1}^\ell$. The positivity in Lemma 13 and (15) give

$$\|\nu_k\|_{\text{TV}} \leq C_n \|\mu_{k-1}\|_{\text{TV}} (1 + L_{k-1})^{n-2} 2^{-L_{k-1}}.$$

The multipliers after $\|\mu_{k-1}\|_{\text{TV}}$ form a summable sequence. Iterating $\|\mu_k\|_{\text{TV}} \leq \|\mu_{k-1}\|_{\text{TV}} + \|\nu_k\|_{\text{TV}}$ therefore yields

$$(20) \quad \sup_k \|\mu_k\|_{\text{TV}} < \infty, \quad \|\nu_k\|_{\text{TV}} \leq C_n R_k^{2n-4} e^{-c_0 R_k^2}, \quad c_0 = \frac{\log 2}{4}.$$

Enlarging C_n includes the first shell. In particular $\mu = \delta_0 + \sum_{k \geq 1} \nu_k$ is a finite signed measure.

Step 3. Control close to each sphere. For probability surface measure σ on a sphere of dimension $n-2$ and radius η , set $d = \text{dist}(X, \text{supp } \sigma) > 0$. Direct integration gives

$$(21) \quad \int |X - Y|^{2-n} d\sigma(Y) \leq C_n \eta^{2-n} (1 + \log^+(\eta/d)), \quad \log^+ t = \max\{0, \log t\}.$$

To verify this, first consider $d \geq \eta/2$, when the integral is bounded by $C_n \eta^{2-n}$. Otherwise the transverse radius of X is comparable to η . In polar coordinates centered at the nearest point, the only singular integral is bounded by a constant times $\int_0^1 t^{n-3} (d^2 + \eta^2 t^2)^{-(n-2)/2} dt$. Splitting at $t = d/\eta$ proves (21). The argument includes $n=3$, where the sphere is a circle.

On Ω , the distance to $\Sigma_{k,j}$ exceeds ε_k . Equations (9), (7), and (21) imply

$$(22) \quad \sup_{X \in \Omega} \left| \int |X - Y|^{2-n} d\nu_k(Y) \right| \leq C_n R_k^{n+4} \|\nu_k\|_{\text{TV}}.$$

Thus the series

$$(23) \quad H(X) = |X|^{2-n} + \sum_{k \geq 1} \int |X - Y|^{2-n} d\nu_k(Y)$$

converges locally with all derivatives away from its source sets. For the tail this follows also by differentiating the kernel on a fixed compact set separated from the remote shells. It defines a harmonic function on Ω .

Step 4. Decay between successive shells. Suppose $R_k \leq |X| < R_{k+1}$ and $k \geq 2$. For the measure μ_{k-1} , integrate the expansion (14) shell by shell. Equation (19) annuls the terms of degree at most L_{k-1} . Because $R_{k-1}/|X| \leq 1/2$, (15) and (20) give

$$(24) \quad \left| \int |X - Y|^{2-n} d\mu_{k-1}(Y) \right| \leq C_n (1 + |X|)^{2n} e^{-(\log 2)|X|^2/16}.$$

Indeed, $L_{k-1} = R_k^2/4 > |X|^2/16$, and the remaining coefficient sum is a polynomial times $2^{-L_{k-1}}$. For the outer shells, (20) and (22) yield

$$(25) \quad \sum_{j \geq k} \left| \int |X - Y|^{2-n} d\nu_j(Y) \right| \leq C_n \sum_{j \geq k} R_j^{3n} e^{-c_0 R_j^2} \leq C_n e^{-(\log 2)|X|^2/32}.$$

In the last step, absorb the fixed polynomial into half the exponent, use $R_j = 2^{j-k} R_k$, and then $|X| < 2R_k$. Equations (23), (24), and (25) prove (17) for large $|X|$. On the remaining bounded part, (22) and (20) give a uniform bound, so the constant can be enlarged.

Step 5. Nonvanishing. The unit sphere lies in Ω . Inside it the only source in (23) is the unit atom at the origin; all shell potentials are harmonic in a neighborhood of the closed unit ball. The flux of ∇H across the unit sphere is therefore $-(n-2)|\mathbb{S}^{n-1}| \neq 0$. Hence $H \neq 0$. $\square$

5. AN UPPER BOUND FOR ALL HARMONIC FUNCTIONS

The potential in Proposition 14 is rotationally invariant. The upper bound must not make that assumption. We first prove two planar lemmas and then apply them to every angular coefficient of an arbitrary harmonic function.

5.1. An entire function with the prescribed nodes. Define

$$(26) \quad E(z) = \begin{cases} \prod_{k \geq 1} (1 + (z/R_k)^{2N_k}), & n \text{ odd}, \\ \prod_{k \geq 1} (1 - (z/R_k)^{2(N_k+1)}), & n \text{ even}. \end{cases}$$

The product converges uniformly on compact sets. Its nonreal zeros are exactly $\Lambda \cup \Lambda^\#$, all simple. In even dimensions there are also real zeros at $\pm R_k$, which will not enter the half-plane argument. Set $d(z) = \text{dist}(z, \Lambda \cup \Lambda^\#)$ and $\rho(z) = \min\{1/8, d(z)/4\}$.

Lemma 15. *The function in (26) satisfies*

$$(27) \quad |E(z)| \leq e^{10(1+|z|^2)}, \quad \frac{|E(z)|}{\rho(z)} \leq Ce^{20|z|^2} \quad (z \notin \Lambda \cup \Lambda^\#).$$

Proof. The degree p_k of the k th factor lies between $4R_k^2$ and $6R_k^2$. This proves compact convergence by comparison with $\sum_k (R/R_k)^{4R_k^2}$ for each fixed R and all sufficiently large k .

Suppose $R_m \leq R < 2R_m$. For the inner factors,

$$\sum_{k \leq m} \log(1 + (R/R_k)^{p_k}) \leq m \log 2 + 6 \sum_{k \leq m} R_k^2 \log(R/R_k).$$

Put $q = R/R_m \in [1, 2)$. The last sum, without the factor 6, is at most $R_m^2[(4/3) \log q + (4/9) \log 2]$, and hence at most $(16/9)(\log 2)R^2$. Also $m \log 2 \leq R^2/2$. The first outer factor contributes at most $\log 2$, and the remaining factors contribute less than 1, since $R/R_{m+j} \leq 2^{1-j}$ for $j \geq 2$. These bounds imply the first inequality in (27). If $R < R_1$, use the bound $\log 2$ for the first factor and the same convergent outer sum.

If $d(z) \geq 1/2$, the second inequality follows from the first. Otherwise let λ be a closest nonreal zero. Integrate E' on the segment from λ to z and apply the Cauchy estimate with radius one at each point of that segment. With $R = |z|$ this gives $|E(z)|/\rho(z) \leq 4e^{10(1+(R+2)^2)}$. The inequality $10(1+(R+2)^2) \leq 20R^2 + 90$ completes the proof. $\square$

5.2. A Cauchy-transform factorization. The local similarity principle belongs to the classical theory of generalized analytic functions; see Vekua [Vek62, Part One]. We include a direct proof because the growth estimate at infinity, rather than merely local factorization, is needed later. We use $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ in this subsection.

Lemma 16. *Let $G \subset \mathbb{C}$ be open and $f \in C^1(G)$. Suppose $\bar{\partial}f = Af + B\bar{f}$ on G , where A, B are measurable and $|A| + |B| \leq M$. There is a continuous function s on $\mathbb{C}$ such that $e^{-s}f$ is holomorphic on G and*

$$(28) \quad |s(z)| \leq CM(1 + |z| \log(2 + |z|)).$$

Proof. Set $\alpha = A + B\bar{f}/f$ where $f \neq 0$, set $\alpha = 0$ at zeros of f , and extend it by zero outside G . The assumed equation implies $\bar{\partial}f = \alpha f$ also at its zeros, and $|\alpha| \leq M$. Define

$$(29) \quad s(z) = \frac{1}{\pi} \int_{\mathbb{C}} \alpha(\zeta) \left(\frac{1}{z - \zeta} + \mathbf{1}_{\{|\zeta| \geq 1\}} \left(\frac{1}{\zeta} + \frac{z}{\zeta^2} \right) \right) dA(\zeta).$$

The integral is absolutely convergent. In the region $|\zeta| > 2|z| + 2$, the kernel in parentheses is $O(|z|^2/|\zeta|^3)$; in the complementary disk its singularity at $\zeta = z$ is integrable. Integration of the three terms in that disk gives (28): the only logarithmic loss is $|z| \int_1^{2|z|+2} t^{-1} dt$.

The same decomposition, together with local integrability of the Cauchy kernel, proves continuity. Its fundamental-solution identity $\bar{\partial}(1/(\pi z)) = \delta_0$ gives $\bar{\partial}s = \alpha$ in distributions; the two correction terms in (29) are holomorphic in z . One may verify this identity first on compactly supported test functions with a truncated integral and then pass to the limit using the displayed kernel bound.

For completeness, the multiplication by e^{-s} does not require an unproved classical derivative of s . Let s_ε be a smooth convolution approximation. Locally, $s_\varepsilon \rightarrow s$ uniformly and $\bar{\partial}s_\varepsilon = \alpha_\varepsilon \rightarrow \alpha$ in L^1 . On a compact subset of G ,

$$\bar{\partial}(e^{-s_\varepsilon}f) = e^{-s_\varepsilon}(\alpha - \alpha_\varepsilon)f \rightarrow 0 \quad \text{in } L^1.$$

Thus $\bar{\partial}(e^{-s}f) = 0$ distributionally. A further convolution on small disks, followed by local uniform convergence, shows that the continuous function $e^{-s}f$ is holomorphic. $\square$

5.3. Uniqueness on a perforated half-plane. There are arbitrarily large numbers $b > 1$ for which the line $\text{Im } z = b$ misses every disk $B(\lambda_{k,j}, 4\sqrt{\varepsilon_k})$. Indeed, $\sum_k 8N_k\sqrt{\varepsilon_k} < 1$, by (7); the total length of all their vertical projections is less than one. Fix any such b and put

$$(30) \quad \mathcal{P}_b = \{\text{Im } z > b\} \setminus \bigcup_{\eta_{k,j} > b} \overline{B(\lambda_{k,j}, 2\varepsilon_k)}.$$

This domain is connected: compact paths in the half-plane can be modified around the finitely many disjoint disks they meet.

Lemma 17. *If F is bounded and holomorphic on $\mathcal{P}_b$ and $F(z) = O(e^{-\beta|z|^2})$ at infinity there for some $\beta > 0$, then $F = 0$.*

Proof. Step 1. A harmonic comparison function. For a node $\lambda = \lambda_{k,j}$ with $\eta = \text{Im } \lambda > b$, write $\tau_\lambda = \eta - b$ and $\lambda^* = \text{Re } \lambda + i(2b - \eta)$. Set

$$g_\lambda(z) = \log \frac{|z - \lambda^*|}{|z - \lambda|}, \quad m_\lambda = \min_{|z - \lambda| = 2\varepsilon_k} g_\lambda(z), \quad c_\lambda = \frac{\tau_\lambda + 1}{m_\lambda}.$$

The function g_λ is positive and harmonic in the half-plane away from its pole, and zero on its boundary line. Since $\tau_\lambda \geq 4\sqrt{\varepsilon_k}$, a direct minimization gives $m_\lambda = \log((2\tau_\lambda - 2\varepsilon_k)/(2\varepsilon_k)) \geq R_k^6/2$. Consequently

$$(31) \quad 0 < c_\lambda \leq CR_k^{-5}, \quad \sum_{\eta_{k,j} > b} c_{\lambda_{k,j}} < \infty.$$

The second assertion follows from $N_k \leq 3R_k^2$. Define

$$(32) \quad q_b(z) = \text{Im } z - b - \sum_{\eta_{k,j} > b} c_{\lambda_{k,j}} g_{\lambda_{k,j}}(z).$$

On a fixed compact set separated from the poles, $g_\lambda(z) = O(\tau_\lambda/|\lambda|^2)$ for remote nodes. Thus (31) ensures local uniform convergence and harmonicity, including continuous extension to each finite boundary piece. The defining coefficients give

$$(33) \quad q_b = 0 \text{ on } \text{Im } z = b, \quad q_b \leq 0 \text{ on every hole boundary}, \quad q_b(z) \leq \text{Im } z - b \text{ on } \mathcal{P}_b.$$

On a hole boundary, its own term is at least $\tau_\lambda + 1$, whereas $\text{Im } z - b \leq \tau_\lambda + 2\varepsilon_k$; all other subtracted terms are nonnegative.

Step 2. Positivity at some interior points. Let $Y_m = 3R_m/2$. For every node with $|\lambda| = R_k$,

$$|iY_m - \lambda| \geq |Y_m - R_k| \geq (Y_m + R_k)/7.$$

Since $\tau_\lambda \leq R_k$, this implies $g_\lambda(iY_m) \leq \log 15$. Equations (31) and (32) yield $q_b(iY_m) \geq Y_m - b - (\log 15) \sum c_\lambda > 0$ for large m . The set $\{q_b > 0\}$ therefore contains a nonempty open subset of $\mathcal{P}_b$.

Step 3. The maximum principle. For a fixed $t > 0$, the function $|F|^2 e^{2tq_b}$ is subharmonic; locally it is the modulus squared of a holomorphic function, by a local harmonic conjugate of q_b . The circles $|z| = Y_m$ avoid all holes. On $\mathcal{P}_b \cap B(0, Y_m)$, the upper boundary limits on the line and hole circles do not exceed $\|F\|_\infty^2$, by (33). On the outer circle they are at most $C^2 e^{-2\beta Y_m^2 + 2tY_m}$ for large m . The subharmonic maximum principle and $m \rightarrow \infty$ give $|F(z)|^2 e^{2tq_b(z)} \leq \|F\|_\infty^2$. At a point where $q_b > 0$, let $t \rightarrow \infty$. Then $F = 0$ on a nonempty open set, and hence on the connected domain $\mathcal{P}_b$. $\square$

5.4. All spherical harmonic coefficients. We can now state the tail estimate needed for changing the topology of the boundary.

Proposition 18. *Let s_m be as in (12). If $V \in \mathcal{H}(\Omega \cap \{|X| > s_m\})$ and $V(X) = O(e^{-a|X|^2})$ at infinity for some $a > 20$, then $V = 0$.*

Proof. Step 1. Angular projection. Choose $20 < a_1 < a$ and then choose b as in (30), larger by two than s_m and the radius beyond which the assumed estimate for V holds. Fix a real spherical harmonic Y of degree ℓ on $\mathbb{S}^{n-2}$ and put $W(x, r) = \int_{\mathbb{S}^{n-2}} V(x, r\omega) Y(\omega) d\sigma(\omega)$, where σ is probability surface measure. The polar-coordinate Laplacian gives

$$W_{xx} + W_{rr} + \frac{n-2}{r} W_r - \frac{\ell(\ell+n-3)}{r^2} W = 0.$$

For $v = r^{-\ell} W$ and $f = v_x - iv_r$, it follows that

$$(34) \quad \bar{\partial} f = \frac{2\ell + n - 2}{4ir} (f - \bar{f}).$$

All functions here are defined on the meridian half-plane outside the original holes, and in particular on $\mathcal{P}_b$. Equation (34) has bounded coefficients there. Apply Lemma 16 to obtain s with (28) (with a constant depending on n, ℓ, b) such that $e^{-s} f$ is holomorphic on $\mathcal{P}_b$.

Step 2. A bound uniform up to the smaller holes. For $z = x + ir \in \mathcal{P}_b$ and every $\omega \in \mathbb{S}^{n-2}$, (11) implies $B((x, r\omega), \rho(z)) \subset \Omega \cap \{|X| > s_m\}$. Indeed the holes in $\mathcal{P}_b$ have twice their original radii, $\rho(z) \leq d(z)/4$, and the chosen horizontal line stays more than one unit from the original tail boundary. The assumed Gaussian estimate holds on these balls. Since $\rho(z) \leq 1/8$, displacement within a ball can be absorbed by replacing a with a_1 . Lemma 9, applied before angular integration, gives

$$(35) \quad |f(z)| \leq \frac{C_{n,\ell}}{\rho(z)} e^{-a_1|z|^2} \quad (z \in \mathcal{P}_b).$$

The derivative of the factor $r^{-\ell}$ introduces only an additional bounded coefficient because $r > b > 1$.

Step 3. Vanishing of each coefficient. The function $J = Ee^{-s}f$ is holomorphic on $\mathcal{P}_b$. Lemma 15, (28), and (35) give

$$|J(z)| \leq C_{n,\ell} \exp(-(a_1 - 20)|z|^2 + C_{n,\ell}|z| \log(2 + |z|)).$$

Since $a_1 > 20$, this is a bounded function with a positive Gaussian decay rate. Lemma 17 yields $J = 0$. The function E has no zeros on $\mathcal{P}_b$, so $f = 0$ and v is constant. Thus $W = r^\ell v$. Since $r \geq b > 1$, its Gaussian decay along any sequence of points of $\mathcal{P}_b$ tending to infinity forces this constant to be zero.

This holds for every Y . Lemma 11 therefore gives $V = 0$ on the nonempty open subset whose meridian points lie in $\mathcal{P}_b$. Unique continuation for harmonic functions, together with (12), proves the claim on the entire tail. $\square$

6. JOINING THE OBSTACLES

We next arrange connected boundary without presuming that a small boundary perturbation preserves harmonic decay rates. The argument first finds a suitable open neighborhood of a connected closed set, and only then chooses the smooth boundary within that neighborhood.

6.1. A fixed connected set and a reserved open tube. In the two-dimensional slice $\{(x, r, 0, \dots, 0) : r \geq 0\}$, let $\mathcal{T}$ be the union of the vertical ray $(0, r, 0, \dots, 0)$, $r \geq 1/2$, and the arcs

$$\{(R_k \cos \theta, R_k \sin \theta, 0, \dots, 0) : \theta_{k,1} \leq \theta \leq \theta_{k,N_k}\}, \quad k \geq 1.$$

This is a locally finite embedded graph. After contracting each obstacle that it meets to a vertex, it is a tree, with finite valence at each vertex. Put $K_0 = K_b \cup \mathcal{T}$ and reserve the open set

$$(36) \quad \mathcal{U} = \left\{ (x, y) : x > 1, \quad |y| < \frac{1}{100(n+1)(1+x)} \right\}.$$

The set K_0 is closed and connected. The tube in (36) will be kept in every final domain. Its intersection with a bounded set meets only finitely many arcs and obstacles. Moreover, $\overline{\mathcal{U}} \cap K_0 = \emptyset$. To verify the last assertion, the ball and the vertical ray are immediate. On a shell with $x \geq R_k/2$, (9) bounds the transverse radius of each arc point below by $1/(3R_k)$, whereas the tube radius is at most $2/[100(n+1)R_k]$. If $x < R_k/2$ and $x > 1$, the shell's transverse radius is at least $\sqrt{3}R_k/2$. The obstacle thickness ε_k is smaller than these margins.

Every set $\mathcal{U} \cap \{|X| > R\}$ is connected. For a point (x_0, y_0) in it, increase x and set $y(x) = y_0(1+x_0)/(1+x)$. The path stays in $\mathcal{U}$ and its squared norm has derivative $2x - 2|y(x)|^2/(1+x) > 0$. After reaching $x > R+1$, join the point to the axis. Such paths connect every point to the same tail of the axis.

6.2. A uniform exclusion obtained by compactness.

Lemma 19. *Fix $a > 20$, $M, R > 0$, and $p \in \mathcal{U}$ with $|p| > R$. Choose $N = s_m > \max\{R, |p|\} + 2$. There is $\varepsilon > 0$ for which no open set G and $w \in \mathcal{H}(G)$ satisfy all of*

$$(37) \quad \begin{aligned} & \mathcal{U} \subset G \subset \mathbb{R}^n \setminus K_0, \\ & \{|X| > N, \text{ dist}(X, K_0) > \varepsilon\} \subset G, \\ & w(p) = 1, \quad |w(X)| \leq Me^{-a|X|^2} \quad (X \in G, |X| > R). \end{aligned}$$

Proof. If the assertion fails, take $\varepsilon_j \downarrow 0$ and corresponding G_j, w_j . On every compact subset of the fixed open set

$$F = (\mathcal{U} \cap \{|X| > R\}) \cup (\{|X| > N\} \setminus K_0),$$

the functions w_j are eventually defined and uniformly bounded. Apply Lemma 9 on an exhaustion and take a diagonal subsequence. It converges locally uniformly to a harmonic function w_∞ on F , with $w_\infty(p) = 1$ and the Gaussian bound in (37).

Within $\Omega \cap \{|X| > N\}$, the omitted set $\mathcal{T} \cap \Omega$ is relatively closed and locally a finite union of smooth arcs. The bound makes w_∞ locally bounded near that set. Lemma 10 extends it harmonically to $\Omega \cap \{|X| > N\}$, retaining the bound by continuity. Proposition 18 makes this extension zero. It agrees with w_∞ on the overlap with the connected set $\mathcal{U} \cap \{|X| > R\}$. Unique continuation on that set gives $w_\infty(p) = 0$, a contradiction. $\square$

The quantifiers in this lemma allow us to choose one neighborhood that excludes all too rapidly decaying harmonic functions.

Lemma 20. *There is an open set $O \supset K_0$, disjoint from $\overline{U}$, such that every connected open set G with*

$$(38) \quad \mathbb{R}^n \setminus O \subset G \subset \mathbb{R}^n \setminus K_0$$

satisfies $(0, a_*] \subset \mathcal{E}(G) \subset (0, 20]$.

Proof. Choose a countable dense set $P \subset \mathcal{U}$. Enumerate all quadruples (a_j, M_j, R_j, p_j) for which $a_j > 20$ is rational, M_j, R_j are positive integers, $p_j \in P$, and $|p_j| > R_j$. Choose strictly increasing radii $N_j = s_{m_j}$ with $N_j > \max\{R_j, |p_j|\} + 2$. Lemma 19 provides a number $\varepsilon_j > 0$ for each quadruple.

Choose a smooth positive function δ on $[0, \infty)$ such that $\delta(t) < \varepsilon_j/2$ for $t \geq N_j$ and such that

$$(39) \quad O = \{X : \text{dist}(X, K_0) < \delta(|X|)\} \text{ is disjoint from } \overline{U}.$$

This choice is possible without a uniform separation of the two closed sets. On each compact annulus the distance from $\overline{U}$ to K_0 has a positive lower bound, and only finitely many of the constraints $N_j \leq t$ apply. First choose a positive continuous minorant of these local bounds, and then a smaller smooth positive function using a locally finite partition of unity on the half-line. This proves (39).

Let G satisfy (38). The restriction to G of the function H in Proposition 14 is not zero, by connectedness of Ω and unique continuation. It obeys (17), proving the lower inclusion.

If some nonzero $V \in \mathcal{H}(G)$ had rate $a' > 20$, choose a rational a with $20 < a < a'$, and an integer R beyond which its estimate holds. Since $\mathcal{U} \subset G$, unique continuation supplies $p \in P$, $|p| > R$, with $V(p) \neq 0$. Normalize $w = V/V(p)$ and choose an integer M large enough for the bound with rate a . These data occur at an index j in the enumeration. If $|X| > N_j$ and $\text{dist}(X, K_0) > \varepsilon_j$, then (39) implies $X \notin O$, hence $X \in G$. Thus G, w satisfy (37), contrary to the choice of ε_j . This proves the upper inclusion. $\square$

6.3. A smooth neighborhood with connected boundary. The geometry can be handled directly by tubes with disk cross sections. In particular, it is not necessary to assert a general stability theorem for harmonic functions under smoothing.

Lemma 21. *For the closed set K_0 constructed above and every open neighborhood O of K_0 , there is a closed set K such that*

$$(40) \quad K_0 \subset \text{int } K, \quad K \subset O,$$

∂K is a smooth connected embedded hypersurface, and $\mathbb{R}^n \setminus K$ is connected whenever it is nonempty.

Proof. Step 1. Blocks and tubes. The original obstacles in (10) are smooth compact n -dimensional domains with connected boundaries. Enlarge each slightly, still disjoint from the others and inside O , so that it contains the corresponding original obstacle in its interior. Add small disjoint closed balls at the finitely valent vertices of $\mathcal{T}$ outside these blocks. Subdivide the graph at these balls and at its passages through the enlarged obstacles. Its remaining edges are compact arcs between blocks. Corners on an edge can be smoothed in a small neighborhood; a tube can be chosen to contain both the old arc and its smoothed center curve. All choices are locally finite, since the original graph and the blocks are locally finite.

For each edge choose a thin closed tubular neighborhood with disk cross section $\overline{B}^{n-1}$. At its endpoints it attaches to two disjoint $(n-1)$ -disks in the block boundaries. The number of attachments to a block is finite. A short collar makes the center curve transverse to the attachment disk; distinct tubes can be chosen disjoint away from their blocks. This follows here simply by separating the finitely many compact arcs in each bounded region, with smaller radii chosen on successive regions. Each block and each tube lies in O , contains its part of K_0 in the interior, and the entire family is locally finite. Their union is closed. The graph of blocks and tubes is connected.

Round the attachment corners in smaller pairwise disjoint collars. In local coordinates this replaces the corner of two intersecting half-spaces by a smooth rounded corner. The replacement is a small deformation supported away from K_0 , and can be confined to O . Local finiteness prevents accumulation of these operations at a finite point. The resulting closed set K is an n -dimensional smooth manifold with boundary and satisfies (40).

Step 2. Connectedness of the boundary. Before rounding, the boundary consists of each block boundary with finitely many open attachment disks removed, together with the lateral boundaries $\mathbb{S}^{n-2} \times [0, 1]$ of the tubes. Removing finitely many disjoint open disks from a connected manifold of dimension $n-1 \geq 2$ leaves it connected: paths meeting a disk can be detoured along its connected boundary sphere. Each tube's lateral boundary is connected because $n \geq 3$. These pieces meet according to the connected block graph. Any two pieces can therefore be joined through finitely many other pieces. The whole boundary is connected, and rounding preserves this property.

Step 3. Connectedness of the exterior. Let $D = \mathbb{R}^n \setminus K \neq \emptyset$. Every component of D has a nonempty boundary contained in ∂K ; otherwise that component would be both open and closed in $\mathbb{R}^n$. Near a point of ∂K , smoothness identifies D with one side of a coordinate hyperplane. That local side is connected and belongs to a unique component of D . The sets of boundary points adjacent to the respective components form a partition of ∂K into relatively open sets. Each component gives a nonempty such set. Connectedness of ∂K allows only one of them. Hence D is connected. $\square$

The last argument also explains why a mere union of very thin neighborhoods would not suffice: the connectedness of the resulting boundary has to be checked independently of the analytic estimates.

We now prove Theorem 2.

Proof. Take O from Lemma 20 and choose K by Lemma 21. Set $D = \mathbb{R}^n \setminus K$. The tube $\mathcal{U}$ is contained in D , so D is nonempty and unbounded. Lemma 21 gives a smooth connected boundary and a connected domain. The inclusions (40) imply (38), so Lemma 20 gives the asserted rate bounds. The restriction of the function in Proposition 14 supplies the stated global estimate. Every finite boundary point has a local smooth defining function and a local exterior ball, and is therefore regular for the Laplacian's Dirichlet problem. $\square$

Only the scaling argument remains for the heat equation statement.

We now prove Corollary 3.

Proof. For the domain in Theorem 2, put $D_- = \sqrt{a_*} D$ and $D_+ = \sqrt{40} D$. Equation (5) and the attained lower rate imply $\text{II}(D_-)$. The same identity and the upper bound 20 imply $\neg \text{II}(D_+)$: otherwise $40 \in \mathcal{E}(D)$. Lemma 8, with $\lambda = \sqrt{40/a_*} > 1$, gives $\text{I}(D_-) \Rightarrow \text{I}(D_+)$. If $\text{I}(D_-)$ fails, then D_- violates the proposed implication from harmonic decay to a temperature. If it holds, D_+ violates the converse. Both dilates retain smooth connected boundary. $\square$

7. ANALYTIC PRELIMINARIES

For a plane domain G , write $\delta_G(z) = \text{dist}(z, \partial G)$. In two dimensions the real gradient estimate (6) is equivalently

$$(41) \quad |v_x(z_0) - iv_y(z_0)| \leq \frac{C_0}{r} \sup_{z \in B(z_0, r)} |v(z)| \quad (B(z_0, r) \subset G).$$

The constant C_0 is absolute.

We only need the following form of the classical Koebe estimate. The cited corollary states it for arbitrary univalent functions on $\mathbb{D}$.

Lemma 22 ([Kar25, Corollary 2.1, p. 6 of 26]). *If $\psi : \mathbb{D} \rightarrow G$ is conformal, then*

$$(42) \quad \delta_G(\psi(\zeta)) \geq \frac{1}{4}(1 - |\zeta|)|\psi'(\zeta)| \quad (\zeta \in \mathbb{D}).$$

The boundary correspondence will be used only to locate the unique boundary point mapped to infinity. Its spherical version follows from the bounded version by a Möbius transformation.

Lemma 23 ([Pom92, Carathéodory theorem, p. 18]). *A conformal map from $\mathbb{D}$ onto an unbounded Jordan domain extends to a homeomorphism of the closures in $\widehat{\mathbb{C}}$.*

We next record an elementary area estimate. This is the point at which univalence, rather than holomorphy alone, enters the proof of Theorem 4.

Lemma 24. *Let f be univalent on $\mathbb{D}$ and let $p > 0$. If $\int_{1/2}^1 M(r, f)^p dr < \infty$, then $f \in H^p$.*

Proof. The function $m_p(\cdot, f)$ is locally absolutely continuous. The distributional identity $\Delta|f|^p = p^2|f|^{p-2}|f'|^2$ and Green's formula give, for almost every $0 < r < 1$,

$$\frac{d}{dr} m_p(r, f) = \frac{p^2}{2\pi r} \int_{|\zeta| < r} |f(\zeta)|^{p-2} |f'(\zeta)|^2 dA(\zeta).$$

At a zero of f the integrand is locally integrable for every $p > 0$; the identity also follows by regularizing $|f|^p$. Since f is univalent, the area change of variables has multiplicity one. Thus

$$(43) \quad \frac{d}{dr} m_p(r, f) = \frac{p^2}{2\pi r} \int_{f(r\mathbb{D})} |z|^{p-2} dA(z) \leq \frac{p^2}{2\pi r} \int_{|z| < M(r, f)} |z|^{p-2} dA(z) = \frac{p}{r} M(r, f)^p.$$

Integration of (43) from $1/2$ to r proves the assertion. $\square$

For later use we include the harmonic-majorant criterion at a fixed exponent. The characterization of Hardy classes by harmonic majorants is classical; see Kim and Sugawa [KS11, p. 292]. The proof below keeps the endpoint explicit and also gives its exponential formulation.

Proposition 25. *Let $D \subsetneq \mathbb{C}$ be simply connected, let $\phi : \mathbb{D} \rightarrow D$ be conformal, and let $q > 0$. The following conditions are equivalent.*

- (i) $\phi \in H^q$.
- (ii) There is $H \in \mathcal{H}(D)$ such that $H(z) \geq |z|^q$ on D .
- (iii) There are a nonzero holomorphic function f on D and constants $a, C > 0$ such that $|f(z)| \leq Ce^{-a|z|^q}$ on D .
- (iv) For every $A > 0$ there is a zero-free holomorphic function f_A on D such that $|f_A(z)| \leq e^{-A|z|^q}$ on D and $\operatorname{Re} f_A \not\equiv 0$.

Proof. Step 1. The harmonic majorant. Assume (i). For $0 < r < 1$, let P_r be the harmonic function on $r\mathbb{D}$ with boundary values $|\phi|^q$. Subharmonicity gives $P_r \geq |\phi|^q$. If $r < s$, the maximum principle on $r\mathbb{D}$ gives $P_r \leq P_s$ there. Moreover, $P_r(0) = m_q(r, \phi)$ is bounded independently of r . Harnack's inequality and monotone convergence therefore give a finite harmonic limit P on $\mathbb{D}$, with $P \geq |\phi|^q$. The function $H = P \circ \phi^{-1}$ proves (ii). Conversely, if (ii) holds, the mean-value property for $H \circ \phi$ gives $m_q(r, \phi) \leq H(\phi(0))$, which proves (i).

Step 2. The exponential functions. Assume (ii), choose $z_0 \in D$, and let $\tilde{H}$ be a harmonic conjugate of H , normalized by $\tilde{H}(z_0) = 0$. Simple connectivity ensures its existence. Define

$$(44) \quad f_A(z) = \exp\{-A(H(z) + i\tilde{H}(z))\}.$$

Then f_A is zero-free, $|f_A| = e^{-AH} \leq e^{-A|z|^q}$, and $\operatorname{Re} f_A(z_0) = e^{-AH(z_0)} > 0$. This proves (iv), which implies (iii).

Step 3. Recovering the endpoint. Assume (iii), and write $f \circ \phi(\zeta) = \zeta^m k(\zeta)$, where $m \geq 0$ is an integer and $k(0) \neq 0$. The submean inequality for $\log |k|$ gives, for $1/2 \leq r < 1$,

$$\log |k(0)| - m \log 2 \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(\phi(re^{i\theta}))| d\theta \leq \log C - a m_q(r, \phi).$$

The possible zeros on a circle cause only integrable logarithmic singularities. The resulting uniform bound for $m_q(r, \phi)$ proves (i). $\square$

Proposition 25 makes H^q membership independent of the Riemann map. Its global holomorphic bound must not be confused with the asymptotic real-harmonic bound (3).

8. DECAY ON JORDAN DOMAINS

We first work under a global estimate. A translation placing an omitted point at the origin allows boundary distance to be compared with Euclidean distance. Localization will subsequently remove the global hypothesis.

Lemma 26. *Let $G \subsetneq \mathbb{C}$ be unbounded and simply connected, with $0 \notin G$. Suppose $h \in \mathcal{H}(G) \setminus \{0\}$ and $|h(z)| \leq Ce^{-a|z|^q}$ for every $z \in G$, where $a, q > 0$. For every conformal map $\psi : \mathbb{D} \rightarrow G$ there is $C_1 > 0$ such that*

$$(45) \quad M(r, \psi)^q \leq C_1 \frac{\log(e/(1-r))}{1-r} \quad (1/2 \leq r < 1).$$

Proof. Step 1. The complex gradient. Put $\delta(z) = \delta_G(z)$. Since $0 \notin G$, we have $\delta(z) \leq |z|$. For $\xi \in B(z, \delta(z)/2)$, $|\xi| \geq |z|/2$. Apply (41) on this disk. With $g = h_x - ih_y$ and $b = a2^{-q} > 0$, it gives

$$(46) \quad |g(z)| \leq \frac{C_2}{\delta(z)} e^{-b|z|^q} \quad (z \in G).$$

The function g is holomorphic and not identically zero. Indeed, $g \equiv 0$ would make h constant, and decay on the unbounded domain would then force $h = 0$.

The pullback $K = (g \circ \psi)\psi'$ is a nonzero holomorphic function on $\mathbb{D}$. Substituting the estimate (42) from Lemma 22 into (46), we obtain

$$(47) \quad |K(\zeta)| \leq \frac{C_3}{1-|\zeta|} e^{-b|\psi(\zeta)|^q}.$$

Step 2. A logarithmic integral-mean bound. Write $K(\zeta) = \zeta^m K_0(\zeta)$ with $K_0(0) \neq 0$. For $1/2 \leq r < 1$, the submean inequality gives

$$\log |K_0(0)| - m \log 2 \leq \frac{1}{2\pi} \int_0^{2\pi} \log |K(re^{i\theta})| d\theta.$$

Take logarithms in (47) and integrate. There is a constant C_4 such that

$$(48) \quad m_q(r, \psi) \leq C_4 \log \frac{e}{1-r} \quad (1/2 \leq r < 1).$$

This is a bound with a logarithmic loss, not an H^q estimate.

Step 3. From means to a maximum. For $s = (1+r)/2$, the Poisson majorant of the nonnegative subharmonic function $|\psi|^q$ on $s\mathbb{D}$ gives

$$M(r, \psi)^q \leq \frac{s+r}{s-r} m_q(s, \psi).$$

Since $s-r = (1-r)/2$ and $1-s = (1-r)/2$, (48) proves (45). $\square$

To prove Theorem 4, we localize at the unique boundary point corresponding to infinity. The analytic reflection takes place in the source disk; no analyticity of ∂D is being imposed.

Proof. Step 1. Isolating infinity. Translate a point of $\mathbb{C} \setminus D$ to the origin. Translation preserves membership in every H^p and, after decreasing a if necessary, preserves an estimate of the form (3). Thus we may assume $0 \notin D$.

By Lemma 23, and a rotation of the source disk, we may also assume $\phi(1) = \infty$. Choose R so that (3) holds outside $B(0, R)$. Continuity at 1 in the spherical metric gives $0 < \varepsilon < 1/2$ such that, with $U = \mathbb{D} \cap B(1, \varepsilon)$,

$$(49) \quad \phi(U) \subset D \setminus \overline{B(0, R)}.$$

The restriction of h to $\phi(U)$ is nonzero by uniqueness for harmonic functions. Let $T: \mathbb{D} \rightarrow U$ be conformal, normalized by $T(1) = 1$, and put $\psi = \phi \circ T$. The cap U is a Jordan domain, so this normalization is understood through its boundary homeomorphism. By (49), the pair $\phi(U), h|_{\phi(U)}$ satisfies the global hypotheses of Lemma 26.

Step 2. Comparing the boundary distances in the source. Near 1, both the source boundary and the target boundary of T are arcs of the unit circle. Schwarz reflection therefore extends T holomorphically across that arc by the formula $T(\zeta) = 1/\overline{T(1/\bar{\zeta})}$ on the reflected side. The inverse $S = T^{-1}$ has the same local extension. The extended maps are inverses by the identity theorem; in particular, $S'(1) \neq 0$.

There are $c_1, c_2 > 0$ and a neighborhood V of 1 such that

$$(50) \quad c_1(1-|w|) \leq 1-|S(w)| \leq c_2(1-|w|) \quad (w \in U \cap V).$$

For completeness, write $w = r\xi$ with $|\xi| = 1$ near 1. The boundary map preserves orientation and has nonzero derivative, so $\operatorname{Re}(\xi S'(\xi) \overline{S(\xi)}) = |S'(\xi)| > 0$ on a smaller arc. The identity

$$1 - |S(r\xi)|^2 = \int_r^1 2 \operatorname{Re}(\xi S'(t\xi) \overline{S(t\xi)}) dt$$

then bounds its left-hand side above and below by positive multiples of $1-r$. This proves (50) after shrinking V .

Apply (45) to ψ , and use $\phi(w) = \psi(S(w))$ together with (50). The part of $\mathbb{D}$ away from 1 has bounded image under ϕ , by Lemma 23. We conclude that

$$(51) \quad M(r, \phi)^q \leq C_5 \frac{\log(e/(1-r))}{1-r} \quad (1/2 \leq r < 1).$$

Step 3. The Hardy exponents. If $0 < p < q$, the function $(1-r)^{-p/q} \log(e/(1-r))^{p/q}$ is integrable at $r = 1$. Equation (51) therefore gives $\int_{1/2}^1 M(r, \phi)^p dr < \infty$. Lemma 24 proves $\phi \in H^p$. $\square$

9. THE ENDPOINT ON QUASIDISKS

The additional ingredient is a normalized distortion estimate for a global quasiconformal map. We state the Euclidean form needed below. It follows from the quoted quasi-Möbius estimate by placing its fourth point at infinity.

Lemma 27 ([Bon11, Proposition 3.1(i), p. 571]). *Let $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be quasiconformal with $F(\infty) = \infty$. There is an increasing homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$, with $\eta(0) = 0$, such that for distinct $a, b, c \in \mathbb{C}$,*

$$(52) \quad \frac{|F(a) - F(b)|}{|F(c) - F(b)|} \leq \eta\left(\frac{|a - b|}{|c - b|}\right).$$

Proof. The quasi-Möbius inequality controls the absolute cross-ratio of four points by a homeomorphism applied to the original absolute cross-ratio. In the limit as the fourth point tends to infinity, the cross-ratio becomes $|a - b|/|c - b|$. Its image has the same limit because F fixes infinity. This is (52). $\square$

We prove Theorem 5 by a fixed inward translation in $\mathbb{H}$. The argument constructs a new holomorphic function on D ; it does not assert decay of a harmonic conjugate of the original function.

Proof. Step 1. A shifted complex gradient. Assume (i). Let $F : \mathbb{H} \rightarrow D$ be conformal with the global quasiconformal extension from the definition, and put $V = h \circ F$. This is real harmonic on $\mathbb{H}$. For $w = \xi + iv \in \mathbb{H}$, define the holomorphic function

$$(53) \quad g(w) = (V_\xi - iV_v)(w + i).$$

For $\zeta \in B(w + i, 1/2)$, we have $|\zeta - w| \leq 3/2$. Apply Lemma 27, in the form (52), to the three points $\zeta, w, 0$:

$$|F(\zeta) - F(w)| \leq \eta\left(\frac{3}{2|w|}\right) |F(0) - F(w)|.$$

Since F is a homeomorphism fixing infinity, $|F(w)| \rightarrow \infty$ uniformly as $|w| \rightarrow \infty$. Also $\eta(t) \rightarrow 0$ as $t \downarrow 0$. Thus, for every sufficiently large $|w|$ and every such ζ ,

$$(54) \quad |F(\zeta)| \geq \frac{1}{2}|F(w)|.$$

The comparison is uniform over the disk $B(w + i, 1/2)$.

Step 2. A global holomorphic decay estimate. Apply Lemma 9 to V on $B(w + i, 1/2)$. The hypothesis (3), the definition (53), and (54) give

$$(55) \quad |g(w)| \leq C_1 \exp\{-a2^{-q}|F(w)|^q\}$$

for all sufficiently large $|w|$.

After increasing C_1 , (55) holds for every $w \in \mathbb{H}$. Indeed, for bounded $|w|$ all the disks $B(w + i, 1/2)$ lie in one compact subset of $\mathbb{H}$, whereas $F(w)$ is bounded by continuity of its global extension on a closed half-disk. This argument does not require boundedness of h near a finite point of ∂D .

The function g is not identically zero. Otherwise the holomorphic complex gradient of V vanishes on $\mathbb{H} + i$, and hence on $\mathbb{H}$. The function h would then be constant; its decay on the unbounded domain would force it to be zero. Define $f = g \circ F^{-1}$ on D . It is nonzero and holomorphic, and (55) gives $|f(z)| \leq C_1 e^{-a2^{-q}|z|^q}$ on the whole domain. Proposition 25 proves (ii).

Step 3. Every rate. If (ii) holds, take the functions f_A constructed in (44) in the proof of Proposition 25. Their real parts are nonzero harmonic functions with the bound in (iii). The implication (iii) $\Rightarrow$ (i) is immediate. $\square$

10. GAUSSIAN DECAY AND THE HEAT EQUATION

We first prove Corollary 6, the Gaussian specialization of the quasidisk theorem.

Proof. If $\phi \in H^2$, Theorem 5 with $q = 2$ realizes all positive Gaussian rates, so (2) gives $\gamma(D) = +\infty$. If $\phi \notin H^2$, the same theorem excludes every positive rate, and (2) gives $\gamma(D) = 0$. $\square$

A finite positive value can therefore occur, if at all, only outside the quasidisk class. We now prove Corollary 7, which also determines the Hardy endpoint.

Proof. The inequality $\gamma(D) > 0$ implies the existence of a nonzero harmonic function with some positive Gaussian decay rate. Theorem 4, with $q = 2$, gives $\phi \in H^p$ for every $p < 2$, hence $\mathfrak{h}(D) \geq 2$. If $\phi \in H^2$, Proposition 25 would give $\gamma(D) = +\infty$, contrary to the hypothesis. If $\mathfrak{h}(D) > 2$, then $\phi \in H^p$ for some $p > 2$; the inclusion $H^p \subset H^2$ would give the same contradiction. Thus $\mathfrak{h}(D) = 2$ and $\phi \notin H^2$. Corollary 6 excludes an unbounded quasidisk. $\square$

10.1. The endpoint distinction. The standard sector calculation is included to distinguish membership in every H^p , $p < 2$, from membership in H^2 ; see Kim and Sugawa [KS11, Example 2.2, p. 293] for the sector Hardy number. This example does not establish sharpness of Theorem 4: as shown below, the sector does not satisfy its Gaussian decay hypothesis.

Example 28. Let $Q = \{z \in \mathbb{C} : |\arg z| < \pi/4\}$. The conformal map

$$(56) \quad \phi_*(\zeta) = \left(\frac{1+\zeta}{1-\zeta} \right)^{1/2}$$

uses the square root with positive real part. It belongs to H^p exactly when $0 < p < 2$, and $\gamma(Q) = 0$.

Proof. For r near 1 and $|\theta|$ small, (56) gives

$$|\phi_*(re^{i\theta})|^p \asymp ((1-r)^2 + \theta^2)^{-p/4},$$

where $\asymp$ means upper and lower bounds by fixed positive multiples. Away from $\theta = 0$ the values are uniformly bounded. Integration shows bounded means for $p < 2$, logarithmic divergence for $p = 2$, and divergence for $p > 2$.

For clarity, Q is an unbounded quasidisk. The map $F(w) = e^{-i\pi/4}w^{1/2}$ is conformal from $\mathbb{H}$ onto Q . In polar coordinates, extend it to $\mathbb{C} \setminus \{0\}$ by $F(re^{i\theta}) = r^{1/2}e^{i\alpha(\theta)}$, where

$$\alpha(\theta) = \begin{cases} -\pi/4 + \theta/2, & 0 \leq \theta \leq \pi, \\ \pi/4 + 3(\theta - \pi)/2, & \pi \leq \theta \leq 2\pi. \end{cases}$$

The angular values agree modulo 2π at the endpoints. This is an orientation-preserving homeomorphism with radial and angular stretch ratio at most 3 almost everywhere; its derivatives are locally square integrable at the origin. It is therefore quasiconformal and extends by $F(0) = 0$, $F(\infty) = \infty$. Corollary 6 now gives $\gamma(Q) = 0$. $\square$

A smooth connected boundary does not remove this harmonic obstruction.

Example 29. The domain $D_* = \{x+iy : x > \sqrt{1+y^2}\}$ has a noncompact, connected, real-analytic, strictly convex boundary and satisfies $\gamma(D_*) = 0$.

Proof. The boundary is the graph of $y \mapsto \sqrt{1+y^2}$, whose second derivative is $(1+y^2)^{-3/2} > 0$. Moreover,

$$(57) \quad Q + 1 \subset D_* \subset Q.$$

Suppose a nonzero $h \in \mathcal{H}(D_*)$ has some positive Gaussian decay rate. By (57), the function $v(w) = h(w+1)$ is harmonic on Q , and it is not identically zero by uniqueness for harmonic functions. Since $\operatorname{Re} w > 0$ on Q , we have $|w+1| \geq |w|$. Thus v has a positive Gaussian decay rate on Q , contrary to Example 28. $\square$

10.2. What the harmonic criteria do not decide. A noncompact, connected, smooth embedded boundary in the plane is a properly embedded copy of $\mathbb{R}$. Its one-point compactification is a Jordan curve. Each side is therefore an unbounded Jordan domain, so Theorem 4 applies to the planar boundary condition considered below.

Smoothness alone does not imply the quasidisk condition. For example, the finite subarc of the parabola $\{y^2 + iy : y \in \mathbb{R}\}$ joining $R^2 - iR$ to $R^2 + iR$ has diameter at least R^2 , whereas the endpoints are only $2R$ apart. On the image of $\mathbb{R}$ under a map satisfying (52), every such subarc has diameter at most $2\eta(1)$ times the distance between its endpoints: apply (52) to each intermediate parameter and the two endpoints. Hence the sides of this smooth parabola are not unbounded quasidisks.

The planar specialization of Problem 1 remains separate from the higher-dimensional counterexamples. We repeat it to distinguish it from the harmonic criteria just proved.

Problem 30. Let $D \subset \mathbb{R}^2$ have a noncompact, connected, smooth boundary. Are the following conditions equivalent?

- (i) There exists a bounded temperature $u \not\equiv 0$ on $D \times [0, 1]$ such that $u(z, 1) = 0$ for every $z \in D$.
- (ii) There exists $h \in \mathcal{H}(D) \setminus \{0\}$ such that $h(z) = O(e^{-|z|^2})$ as $z \rightarrow \infty$ in D .

Theorems 4 and 5 concern condition (ii) in Problem 30. They do not characterize condition (i). In particular, a conformal change of variables is not a change between solutions of the usual heat equation. If $U(\zeta, t) = u(\phi(\zeta), t)$, the chain rule gives

$$(58) \quad U_t = |\phi'(\zeta)|^{-2} \Delta_\zeta U.$$

The spatially varying coefficient in (58) prevents a conclusion about backward uniqueness from the conformal harmonic criteria alone. No proof or counterexample for Problem 30 is asserted here, including for the smooth domain in Example 29.

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